COURSE NOTES
FOR

Bachelor Computer Applications
First Semester

MATH-I

as per syllabus of

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Prepared By:

Department of Computer Science
Microtek College of Management & Technology
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BCA-S105 Mathematics

UNIT-I

DETERMINANTS:

Definition, Minors, Cofactors, Properties of Determinants, MATRICES: Definition, Types of Matrices,
Addition, Subtraction, Scalar Multiplication and Multiplication of Matrices, Adjoint, Inverse, Cramers Rule,
Rank of Matrix Dependence of Vectors, Eigen Vectors of a Matrix, Caley-Hamilton Theorem (without proof).

UNIT-II

LIMITS & CONTINUITY:

Limit at a Point, Properties of Limit, Computation of Limits of Various Types of Functions, Continuity at a
Point, Continuity Over an Interval, Intermediate Value Theorem, Type of Discontinuities

UNIT-III

DIFFERENTIATION:

Derivative, Derivatives of Sum, Differences, Product & Quotients, Chain Rule, Derivatives of Composite
Functions, Logarithmic Differentiation, Rolle's Theorem, Mean Value Theorem, Expansion of Functions
(Maclaurin's & Taylor's), Indeterminate Forms, L' Hospitals Rule, Maxima & Minima, Curve Tracing,
Successive Differentiation & Liebnitz Theorem.

UNIT-IV

INTEGRATION:

Integral as Limit of Sum, Fundamental Theorem of Calculus(without proof.), Indefinite Integrals, Methods of
Integration: Substitution, By Parts, Partial Fractions, Reduction Formulae for Trigonometric Functions, Gamma
and Beta Functions(definition).

UNIT-V

VECTOR ALGEBRA:

Definition of a vector in 2 and 3 Dimensions; Double and Triple Scalar and Vector Product and physical
interpretation of area and volume.

Reference Books :

1. B.S. Grewal, "Elementary Engineering Mathematics", 34th Ed., 1998.
2. Shanti Narayan, "Integral Calculus", S. Chand & Company, 1999
3. H.K. Dass, "Advanced Engineering Mathematics", S. Chand & Company, 9th Revised Edition, 2001.
4. Shanti Narayan, "Differential Calculus ", S.Chand & Company, 1998.

UNIT - I
DETERMINANTS

Contents

DETERMINANTS: Definition, Minors, Cofactors, Properties of Determinants,

MATRICES: Definition, Types of Matrices,Addition, Subtraction, Scalar Multiplication and Multiplication of Matrices, Adjoint, Inverse, Cramers Rule,Rank of Matrix Dependence of Vectors, Eigen Vectors of a Matrix, Caley-Hamilton Theorem (without proof).

DETERMINANTS

Def. Let $A=\left[a_{ij}\right]$ be a square matrix of order n. The determinant of A, detA or |A| is defined as follows:

(a) If n=2, $\det A=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$

(b) If n=3, $\det A=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

or $$\det A=a_{11}a_{22}a_{33}+a_{21}a_{32}a_{13}+a_{31}a_{12}a_{23}$$
$$-a_{31}a_{22}a_{13}-a_{32}a_{23}a_{11}-a_{33}a_{21}a_{12}$$

e.g. Evaluate (a) $\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix}$ (b) $\det\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 1 & -2 & -1 \end{bmatrix}$

e.g. If $\begin{vmatrix} 3 & 2 & x \\ 8 & x & 1 \\ 3 & -2 & 0 \end{vmatrix}=0$, find the value(s) of x.

N.B. $\det A=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad =a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

or $=-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}-a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$

or . . . .

$$\textbf{By using} \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

e.g. Evaluate (a) $\begin{vmatrix} 3 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & 2 & 0 \\ 8 & -2 & 1 \\ 3 & 2 & 3 \end{vmatrix}$

PROPERTIES OF DETERMINANTS

(1) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ i.e. $\det(A^T) = \det A.$

(2) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

(3) $\begin{vmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & 0 & c_3 \end{vmatrix} = 0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 0 \end{vmatrix}$

(4) $\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

(5) If $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$, then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(6) $\begin{vmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}$

(7) $\begin{vmatrix} pa_1 & b_1 & c_1 \\ pa_2 & b_2 & c_2 \\ pa_3 & b_3 & c_3 \end{vmatrix} = p\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ pa_2 & pb_2 & pc_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\begin{vmatrix} pa_1 & pb_1 & pc_1 \\ pa_2 & pb_2 & pc_2 \\ pa_3 & pb_3 & pc_3 \end{vmatrix} = p^3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

N.B. (1) $\begin{pmatrix} pa_1 & pb_1 & pc_1 \\ pa_2 & pb_2 & pc_2 \\ pa_3 & pb_3 & pc_3 \end{pmatrix} = p\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

(2) If the order of A is n, then $\det(\lambda A) = \lambda^n \det(A)$

(8) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + \lambda b_1 & b_1 & c_1 \\ a_2 + \lambda b_2 & b_2 & c_2 \\ a_3 + \lambda b_3 & b_3 & c_3 \end{vmatrix}$

N.B. $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \overset{\alpha C_2 + \beta C_3 + C_1}{=\!=\!=\!=\!=\!=\!=\!=} \begin{vmatrix} x_1 + \alpha y_1 + \beta z_1 & y_1 & z_1 \\ x_2 + \alpha y_2 + \beta z_2 & y_2 & z_2 \\ x_3 + \alpha y_3 + \beta z_3 & y_3 & z_3 \end{vmatrix}$

e.g. Evaluate (a) $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix}$, (b) $\begin{vmatrix} 5 & 3 & 7 \\ 3 & 7 & 5 \\ 7 & 2 & 6 \end{vmatrix}$

e.g. Evaluate $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

e.g. Factorize the determinant

$$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

e.g. Factorize each of the following :

(a) $\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 2a^3 & 2b^3 & 2c^3 \\ a^2 & b^2 & c^2 \\ 1-a^3 & 1-b^3 & 1-c^3 \end{vmatrix}$

Multiplication of Determinants.

Let $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $|B| = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$

Then $|A||B| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$

$$= \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix}$$

Properties :

(1) det(AB)=(detA)(detB) i.e. $|AB| = |A||B|$

(2) |A|(|B||C|)=(|A||B|)|C| N.B. A(BC)=(AB)C

(3) |A||B|=|B||A| N.B. AB≠BA in general

(4) |A|(|B|+|C|)=|A||B|+|A||C| N.B. A(B+C)=AB+AC

e.g. Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Minors and Cofactors

Def. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then A_{ij} , the *cofactor* of a_{ij} , is defined by

$$A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},\ A_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},\ \ldots,\ A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Since $|A| = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = +a_{21}A_{21} - a_{22}A_{22} + a_{23}A_{23}$

Theorem. (a) $a_{i1}A_{j1} + a_{i2}A_{j2} + a_{i3}A_{j3} = \begin{cases} \det A & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

(b) $a_{1i}A_{1j} + a_{2i}A_{2j} + a_{3i}A_{3j} = \begin{cases} \det A & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

e.g. $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = \det A,\quad a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$, etc.

e.g.23 Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and c_{ij} be the cofactor of a_{ij} , where $1 \le i, j \le 3$.

(a) Prove that $A\begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} = (\det A)I$

(b) Hence, deduce that $\begin{vmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{vmatrix} = (\det A)^2$

INTRODUCTION : MATRIX / MATRICES

1. A rectangular array of m×n numbers arranged in the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called an m×n *matrix*.

e.g. $\begin{bmatrix} 2 & 3 & 4 \\ 1 & -8 & 5 \end{bmatrix}$ is a 2×3 matrix.

e.g. $\begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ is a 3×1 matrix.

2. If a matrix has m *rows* and n *columns*, it is said to be *order* m×n.

e.g. $\begin{bmatrix} 2 & 0 & 3 & 6 \\ 3 & 4 & 7 & 0 \\ 1 & 9 & 2 & 5 \end{bmatrix}$ is a matrix of order 3×4.

e.g. $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 5 \\ -1 & 3 & 0 \end{bmatrix}$ is a matrix of order 3.

3. $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ **is called a *row matrix* or *row vector*.**

4. $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ **is called a *column matrix* or *column vector*.**

e.g. $\begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ is a column vector of order 3×1.

e.g. $\begin{bmatrix} -2 & -3 & -4 \end{bmatrix}$ is a row vector of order 1×3.

5. If all elements are real, the matrix is called a real matrix.

6. $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ **is called a *square matrix* of order n. And** $a_{11}, a_{22}, \ldots, a_{nn}$ **is called *the principal diagonal.***

e.g. $\begin{bmatrix} 3 & 9 \\ 0 & -2 \end{bmatrix}$ is a square matrix of order 2.

7. **Notation :** $\left[a_{ij}\right]_{m\times n}$, $\left(a_{ij}\right)_{m\times n}$, A , ...

SOME SPECIAL MATRIX.

Def .1 **If all the elements are zero, the matrix is called a *zero matrix* or null matrix, denoted by $O_{m\times n}$.**

e.g. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 2×2 zero matrix, and denoted by O_2.

Def.2 **Let $A=\left[a_{ij}\right]_{n\times n}$ be a square matrix.**

(i) If $a_{ij}=0$ for all i, j, then A is called a zero matrix.

(ii) If $a_{ij}=0$ for all i<j, then A is called a *lower triangular matrix*.

(iii) If $a_{ij}=0$ for all i>j, then A is called a *upper triangular matrix*.

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & & \vdots \\ \vdots & & & & 0 \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \vdots \\ 0 & 0 & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

i.e. Lower triangular matrix Upper triangular matrix

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 4 \end{bmatrix}$ is a lower triangular matrix.

e.g. $\begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix}$ is an upper triangular matrix.

Def.3 **Let $A=\left[a_{ij}\right]_{n\times n}$ be a square matrix. If $a_{ij}=0$ for all $i\neq j$, then A is called a *diagonal matrix*.**

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix.

Def.4 **If A is a diagonal matrix and $a_{11}=a_{22}=\cdots=a_{nn}=1$, then A is called an *identity matrix* or a *unit matrix*, denoted by I_n.**

e.g. $$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ARITHMETICS OF MATRICES.

Def. 5 **Two matrices A and B are equal iff they are of the *same order* and their *corresponding elements are equal*.**

i.e. $$[a_{ij}]_{m\times n} = [b_{ij}]_{m\times n} \quad \Leftrightarrow \quad a_{ij} = b_{ij} \quad \text{for all } i, j.$$

e.g. $$\begin{bmatrix} a & 2 \\ 4 & b \end{bmatrix} = \begin{bmatrix} -1 & c \\ d & 1 \end{bmatrix} \quad \Leftrightarrow \quad a = -1,\ b = 1,\ c = 2,\ d = 4.$$

N.B. $$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}$$

Def.6 Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$. Define $A + B$ as the matrix $C = [c_{ij}]_{m\times n}$ of the same order such that
$c_{ij} = a_{ij} + b_{ij}$ for all i=1,2,...,m and j=1,2,...,n.

e.g. $$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -4 & 3 \\ 2 & -1 & 5 \end{bmatrix} =$$

N.B. 1. $$\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix} \text{ is not defined.}$$

2. $$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} + 5 \text{ is not defined.}$$

Def.7 Let $A = [a_{ij}]_{m\times n}$. Then $-A = [-a_{ij}]_{m\times n}$ and A-B=A+(-B)

e.g.1 If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 & 0 \\ 3 & -1 & 1 \end{bmatrix}$. Find -A and A-B.

Properties of Matrix Addition.

Let A, B, C be matrices of the same order and O be the zero matrix of the same order. Then

(a) A+B=B+A

(b) (A+B)+C=A+(B+C)

(c) A+(-A)=(-A)+A=O

(d) A+O=O+A

Scalar Multiplication.

Let $A=\left[a_{ij}\right]_{m\times n}$ **, k is scalar. Then kA is the matrix** $C=\left[c_{ij}\right]_{m\times n}$ **defined by** $c_{ij}=ka_{ij}$ **,** $\forall$i, j**.**

i.e. $kA=\left[ka_{ij}\right]_{m\times n}$

e.g. If $A=\begin{bmatrix}3 & -2\\ -5 & 6\end{bmatrix}$, then -2A= ;

N.B. **(1)** **-A=(-1)A**

(2) **A-B=A+(-1)B**

Properties of Scalar Multiplication.

Let A, B be matrices of the same order and h, k be two scalars.

Then (a) k(A+B)=kA+kB

(b) (k+h)A=kA+hA

(c) (hk)A=h(kA)=k(hA)

Let $A=\left[a_{ij}\right]_{m\times n}$. The *transpose* of A, denoted by A^T, or A' , is defined by

$$A^T=\begin{bmatrix}a_{11} & a_{21} & \cdots & a_{m1}\\ a_{12} & a_{22} & \cdots & a_{m2}\\ \vdots & & & \vdots\\ a_{1n} & a_{2n} & \cdots & a_{nm}\end{bmatrix}_{n\times m}$$

e.g. $A=\begin{bmatrix}3 & -2\\ -5 & 6\end{bmatrix}$, then $A^T=$

e.g. $A=\begin{bmatrix}3 & 0 & -2\\ 4 & -6 & 1\end{bmatrix}$, then $A^T=$

e.g. $A=[5]$, then $A^T=$

N.B. (1) $I^T=$

(2) $A=\left[a_{ij}\right]_{m\times n}$, then $A^T=$

Properties of Transpose.

Let A, B be two m×n matrices and k be a scalar, then

(a) $(A^T)^T=$

(b) $(A+B)^T=$

(c) $(kA)^T=$

A square matrix A is called a *symmetric matrix* iff $A^T=A$.

i.e. A is symmetric matrix $\Leftrightarrow$ $A^T=A$ $\Leftrightarrow$ $a_{ij}=a_{ji}$ $\forall$i, j e.g. $\begin{bmatrix} 1 & 3 & -1 \\ 3 & -3 & 0 \\ -1 & 0 & 6 \end{bmatrix}$ is a symmetric matrix.

e.g. $\begin{bmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ -1 & 3 & 6 \end{bmatrix}$ is not a symmetric matrix.

A square matrix A is called a *skew-symmetric matrix* iff $A^T = -A$.

i.e. A is skew-symmetric matrix $\Leftrightarrow A^T = -A \Leftrightarrow a_{ij} = -a_{ji} \quad \forall$i, j

Prove that $A = \begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & 5 \\ 1 & -5 & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Is $a_{ii} = 0$ for all i=1,2,...,n for a skew-symmetric matrix?

Matrix Multiplication.

Let $A = \left[a_{ik}\right]_{m\times n}$ and $B = \left[b_{kj}\right]_{n\times p}$. Then the product AB is defined as the m×p matrix $C = \left[c_{ij}\right]_{m\times p}$ where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} .$$

i.e. $$AB = \left[\sum_{k=1}^{n} a_{ik}b_{kj}\right]_{m\times p}$$

e.g.4 Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}_{3\times 2}$ and $B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}_{2\times 3}$. Find AB and BA.

e.g.5 Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}_{3\times 2}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}_{2\times 2}$. Find AB. Is BA well defined?

N.B. In general, AB $\neq$ BA .

i.e. matrix multiplication is not commutative.

Properties of Matrix Multiplication.

(a) (AB)C = A(BC)

(b) A(B+C) = AB+AC

(c) (A+B)C = AC+BC

(d) AO = OA = O

(e) IA = AI = A

(f) k(AB) = (kA)B = A(kB)

(g) $(AB)^T = B^T A^T$.

N.B. **(1)** **Since AB $\neq$ BA ;**
Hence, A(B+C) $\neq$ (B+C)A and A(kB) $\neq$ (kB)A.

(2) $A^2 + kA = A(A + kI) = (A + kI)A$.

(3) $AB - AC = O \Rightarrow A(B - C) = O$

$$\not\Rightarrow A = O \text{ or } B - C = O$$

e.g. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Then $AB - AC = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

But A $\neq$ O and B $\neq$ C,

so $AB - AC = O \not\Rightarrow A = O$ or $B = C$.

Powers of matrices

For any square matrix A and any positive integer n, the symbol A^n denotes $\underbrace{A \cdot A \cdot A \cdots A}_{n \text{ factors}}$.

N.B. **(1)** $(A + B)^2 = (A + B)(A + B)$

$$= AA + AB + BA + BB$$
$$= A^2 + AB + BA + B^2$$

(2) **If** $AB = BA$**, then** $(A + B)^2 = A^2 + 2AB + B^2$

e.g. Let $A = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 & 0 \\ 3 & -1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Evaluate the following :

(a) $(A + 2B)C$ (b) $(AC)^2$

(c) $(B^T - 3C)^T D$ (d) $(-2A)^T B - DD^T$

e.g. (a) Find a 2x2 matrix A such that

$$2A - 3\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\left[A + \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\right].$$

(b) Find a 2x2 matrix $A=\begin{pmatrix}2 & \alpha\\ \beta & \gamma\end{pmatrix}$ such that

$A^T = A$ and $\begin{pmatrix}2 & 1\\ 3 & 0\end{pmatrix}A = A\begin{pmatrix}2 & 1\\ 3 & 0\end{pmatrix}$.

(c) If $\begin{pmatrix}3 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1\\ x\end{pmatrix} = \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}1\\ x\end{pmatrix}$, find the values of x and λ.

e.g. Let $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$. Prove by mathematical induction that

$$A^n = \begin{pmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{pmatrix} \quad \text{for } n = 1,2,\cdots.$$

e.g. (a) Let $A=\begin{pmatrix}a & 1\\ 0 & b\end{pmatrix}$ where $a,b\in R$ and $a\neq b$.

Prove that $A^n = \begin{pmatrix}a^n & \dfrac{a^n-b^n}{a-b}\\ 0 & b^n\end{pmatrix}$ for all positive integers n.

(b) Hence, or otherwise, evaluate $\begin{pmatrix}1 & 2\\ 0 & 3\end{pmatrix}^{95}$.

e.g. (a) Let $A=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$ and B be a square matrix of order 3. Show that if A and B are commutative, then B is a triangular matrix.

(b) Let A be a square matrix of order 3. If for any $x,y,z\in R$, there exists $\lambda\in R$ such that $A\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \lambda\begin{pmatrix}x\\ y\\ z\end{pmatrix}$, show that A is a diagonal matrix.

(c) If A is a symmetric matrix of order 3 and A is nilpotent of order 2 (i.e. $A^2 = O$), then A=O, where O is the zero matrix of order 3.

Properties of power of matrices :

(1) Let A be a square matrix, then $(A^n)^T = (A^T)^n$.

(2) If $AB = BA$, then

(a) $(A+B)^n = A^n + C_1^n A^{n-1}B + C_2^n A^{n-2}B^2 + C_3^n A^{n-3}B^3 + \cdots + C_{n-1}^n AB^{n-1} + B^n$

(b) $(AB)^n = A^n B^n$.

(3) $(A+I)^n = A^n + C_1^n A^{n-1} + C_2^n A^{n-2} + C_3^n A^{n-3} + \cdots + C_{n-1}^n A + C_n^n I$

e.g (a) Let X and Y be two square matrices such that XY = YX.

Prove that (i) $(X+Y)^2 = X^2 + 2XY + Y^2$

(ii) $(X+Y)^n = \sum_{r=0}^{n} C_r^n X^{n-r} Y^r$ for n = 3, 4, 5, ...

(Note: For any square matrix A , define $A^0 = I$.)

(b) By using (a)(ii) and considering $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$, or otherwise, find

$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}^{100}$.

(c) If X and Y are square matrices,

(i) prove that $(X+Y)^2 = X^2 + 2XY + Y^2$ implies XY = YX ;

(ii) prove that $(X+Y)^3 = X^3 + 3X^2Y + 3XY^2 + Y^3$ does NOT implies XY = YX .

(Hint : Consider a particular X and Y, e.g. $X = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$.)

INVERSE OF A SQUARE MATRIX

N.B. (1) If a, b, c are real numbers such that ab=c and b is non-zero, then $a = \frac{c}{b} = cb^{-1}$ and b^{-1} is usually called the multiplicative inverse of b.

(2) If B, C are matrices, then $\frac{C}{B}$ is undefined.

Def. A square matrix A of order n is said to be non-singular or invertible if and only if there exists a square matrix B such that AB = BA = I.The matrix B is called the multiplicative inverse of A, denoted by A^{-1}

i.e. $AA^{-1} = A^{-1}A = I$.

e.g. Let $A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$, show that the inverse of A is $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$.

i.e. $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$.

e.g. Is $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$?

Non-singular or Invertible

Def. If a square matrix A has an inverse, A is said to be *non-singular* or *invertible*. Otherwise, it is called *singular* or *non-invertible*.

e.g. $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ are both non-singular.

i.e. A is non-singular iff A^{-1} exists.

Thm. The inverse of a non-singular matrix is unique.

N.B. **(1)** $I^{-1} = I$, **so I is always non-singular.**

(2) **OA = O ≠ I , so O is always singular.**

(3) **Since AB = I implies BA = I.**

Hence proof of either AB = I or BA = I is enough to assert that B is the inverse of A.

e.g. Let $A = \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$.

(a) Show that $I - 6A + A^2 = O$.

(b) Show that A is non-singular and find the inverse of A.

(c) Find a matrix X such that $AX = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$.

Properties of Inverses

Thm. Let A, B be two non-singular matrices of the same order and λ be a scalar.

(a) $(A^{-1})^{-1} = A$.

(b) A^T is a non-singular and $(A^T)^{-1} = (A^{-1})^T$.

(c) A^n is a non-singular and $(A^n)^{-1} = (A^{-1})^n$.

(d) λA is a non-singular and $(\lambda A)^{-1} = \dfrac{1}{\lambda} A^{-1}$.

(e) AB is a non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.

INVERSE OF SQUARE MATRIX BY DETERMINANTS

Def. The *cofactor matrix* of A is defined as $cofA = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$.

Def. The *adjoint matrix* of A is defined as

$$adjA = (cofA)^T = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}.$$

e.g. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find adjA.

e.g. (a) Let $A = \begin{pmatrix} 1 & 1 & -3 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, find adjA.

(b) Let $B = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & -1 \\ 5 & 1 & -1 \end{pmatrix}$, find adjB.

Theorem. For any square matrix A of order n , A(adjA) = (adjA)A = (detA)I

$$A(adjA) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

Theorem. Let A be a square matrix. If detA ≠ 0 , then A is non-singular

and $A^{-1} = \frac{1}{\det A}(adjA)$**.**

Proof Let the order of A be n , from the above theorem , $\frac{1}{\det A}(AadjA) = I$

e.g. Given that $A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & -1 \\ 5 & 1 & -1 \end{pmatrix}$, find A^{-1}.

e.g. Suppose that the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-singular , find A^{-1}.

e.g. Given that $A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$, find A^{-1}.

Theorem. A square matrix A is non-singular iff detA ≠ 0 .

e.g. Show that $A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ is non-singular.

e.g. Let $A = \begin{pmatrix} x+1 & 2 & x-1 \\ x-1 & 2 & -1 \\ 5 & 7 & -x \end{pmatrix}$, where $x \in R$.

(a) Find the value(s) of x such that A is non-singular.

(b) If x=3 , find A^{-1}.

N.B. A is singular (non-invertible) iff A^{-1} does not exist.

Theorem. A square matrix A is singular iff detA = 0.

Properties of Inverse matrix.

Let A, B be two non-singular matrices of the same order and λ be a scalar.

(1) $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$

(2) $(A^{-1})^{-1} = A$

(3) $(A^T)^{-1} = \left(A^{-1}\right)^T$

(4) $(A^n)^{-1} = \left(A^{-1}\right)^n$ for any positive integer n.

(5) $(AB)^{-1} = B^{-1}A^{-1}$

(6) The inverse of a matrix is unique.

(7) $\det(A^{-1}) = \frac{1}{\det A}$

N.B. $XY = 0 \not\Rightarrow X = 0$ or $Y = 0$

(8) If A is non-singular , then $AX = 0 \Rightarrow A^{-1}AX = A0 = 0$

$$\Rightarrow X = 0$$

N.B. $XY = XZ \not\Rightarrow X = 0$ or $Y = Z$

(9) If A is non-singular , then $AX = AY \Rightarrow A^{-1}AX = A^{-1}AY$

$$\Rightarrow X = Y$$

(10) $(A^{-1}MA)^n = (A^{-1}MA)(A^{-1}MA)\cdots(A^{-1}MA) = A^{-1}M^nA$

(11) If $M = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, then $M^{-1} = \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{pmatrix}$.

(12) If $M = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, then $M^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$ where $n \neq 0$.

e.g. Let $A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -3 & -1 \\ 0 & 13 & 4 \\ 0 & -33 & -10 \end{pmatrix}$ and $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(a) Find A^{-1} and M^5.

(b) Show that $ABA^{-1} = M$.

(c) Hence, evaluate B^5.

e.g. Let $A = \begin{pmatrix} 3 & 8 \\ 1 & 5 \end{pmatrix}$ and $P = \begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix}$.

(a) Find $P^{-1}AP$.

(b) Find A^n, where n is a positive integer.

e.g. (a) Show that if A is a 3x3 matrix such that $A^t = -A$, then detA=0.

(b) Given that $B = \begin{pmatrix} 1 & -2 & 74 \\ 2 & 1 & -67 \\ -74 & 67 & 1 \end{pmatrix}$,

use (a) , or otherwise , to show $\det(I - B) = 0$.
Hence deduce that $\det(I - B^4) = 0$.

e.g. (a) If α , β and γ are the roots of $x^3 + px + q = 0$, find a cubic equation whose roots are α^2, β^2 and γ^2.

(b) Solve the equation $\begin{vmatrix} x & 2 & 3 \\ 2 & x & 3 \\ 2 & 3 & x \end{vmatrix} = 0$.

Hence, or otherwise, solve the equation

$$x^3 - 38x^2 + 361x - 900 = 0.$$

e.g. Let M be the set of all 2x2 matrices. For any $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M$,

define $tr(A) = a_{11} + a_{22}$.

(a) Show that for any A, B, C ∈ M and α, β ∈ R,
(i) $tr(\alpha A + \beta B) = \alpha tr(A) + \beta tr(B)$,
(ii) $tr(AB) = tr(BA)$,
(iii) the equality "$tr(ABC) = tr(BAC)$" is not necessary true.

(b) Let A ∈ M.
(i) Show that $A^2 - tr(A)A = -(\det A)I$,
where I is the 2x2 identity matrix.

(ii) If $tr(A^2) = 0$ and $tr(A) = 0$, use (a) and (b)(i) to show that A is singular and $A^2 = 0$.

(c) Let S, T $\in$ M such that $(ST - TS)S = S(ST - TS)$.Using (a) and (b) or otherwise, show that

$$(ST - TS)^2 = 0$$

Eigenvalue and Eigenvector

Let $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$ and let x denote a 2x1 matrix.

(a) Find the two real values λ_1 and λ_2 of λ with $\lambda_1 > \lambda_2$ such that the matrix equation

(*) $$Ax = \lambda x$$

has non-zero solutions.

(b) Let x_1 and x_2 be non-zero solutions of (*) corresponding to λ_1 and λ_2 respectively. Show that if

$$x_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$$

then the matrix $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is non-singular.

(c) Using (a) and (b), show that $AX = X\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

and hence $A^n = X\begin{pmatrix} \lambda_1{}^n & 0 \\ 0 & \lambda_2{}^n \end{pmatrix} X^{-1}$ where n is a positive integer.

Evaluate $\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}^n$.

Cramer's rule

The Cramer's rule can be used to solve system of algebraic equations.To solve the system, x_1 and x_2 are written under the form:

$$x_1 = \frac{\begin{vmatrix} \boxed{\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}} & \begin{matrix} a_{12} \\ a_{22} \\ a_{23} \end{matrix} & \begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix} \end{vmatrix}}{D}$$

$$x_2 = \frac{\begin{vmatrix} \begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix} & \boxed{\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}} & \begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix} \end{vmatrix}}{D}$$

And the same thing for x_3

When the number of equations exceeds 3, the Cramer's rule becomes impractical because the computation of the determinants is very time consuming.

Example

Solve using the Cramer's rule the following system $\begin{cases} 3x_1 + x_2 = 5 \\ x_1 - 2x_2 = -3 \end{cases}$

The elimination of unknowns

To illustrate this well known procedure, let us take a simple system of equations with two equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 & (1) \\ a_{21}x_1 + a_{22}x_2 = b_2 & (2) \end{cases}$$

Step I. We multiply (1) by a_{21} and (2) by a_{11}, thus

$$\begin{cases} a_{11}a_{21}x_1 + a_{12}a_{21}x_2 = b_1a_{21} \\ a_{11}a_{21}x_1 + a_{11}a_{22}x_2 = b_2a_{11} \end{cases}$$

By subtracting

$$a_{11}a_{22}x_2 - a_{12}a_{21}x_2 = b_2a_{11} - b_1a_{21}$$

Therefore;

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

Step II. And by replacing in the above equations:

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

Note vv Compare the to the Cramer's law… it is exactly the same.

The problem with this method is that it is very time consuming for a large number of equations.

Rank of a Matrix:

Recall:

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

The i'th row of A is

$$row_i(A) = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}, \quad i = 1,2,\ldots,m,,$$

and the j'th column of A is

$$col_j(A) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, \quad j = 1,2,\ldots,n.$$

Definition of row space and column space:

$$span\{row_1(A), row_2(A), \ldots, row_m(A)\},$$

which is a vector space under standard matrix addition and scalar multiplication, is referred to as *the row space*. Similarly,

$$span\{col_1(A), col_2(A), \ldots, col_n(A)\},$$

which is also a vector space under standard matrix addition and scalar multiplication, is referred to as *the column space*.

Definition of row equivalence:

A matrix B is row equivalent to a matrix A if B result from A via elementary row operations.

Example:

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, B_1 = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

Since

$$A = \begin{array}{c} (1) \\ (2) \\ (3) \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{(1)\leftrightarrow(2)} B_1 = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix},$$

$$A = \begin{array}{c} (1) \\ (2) \\ (3) \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{(1)=2*(1)} B_2 = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

$$A = \begin{array}{c}(1)\\(2)\\(3)\end{array}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} \xrightarrow{(2)=(2)-(1)} B_3 = \begin{bmatrix}1 & 2 & 3\\3 & 3 & 3\\7 & 8 & 9\end{bmatrix},$$

B_1, B_2, B_3 are all ***row equivalent*** to A.

Important Result:

If A and B are two $m \times n$ row equivalent matrices, then the row spaces of A and B are equal.

How to find the bases of the row and column spaces:

Suppose A is a $m \times n$ matrix. Then, the bases of the row and column spaces can be found via the following steps.

Step 1:

Transform the matrix A to the matrix in reduced row echelon form.

Step 2:

- The nonzero rows of the matrix in reduced row echelon form form a basis of the row space of A.
- The columns corresponding to the ones containing the leading 1's form a basis. For example, if $n = 6$ and the reduced row echelon matrix is

$$\begin{bmatrix}1 & \times & \times & \times & \times & \times\\0 & 0 & 1 & \times & \times & \times\\0 & 0 & 0 & 1 & \times & \times\\0 & 0 & 0 & 0 & 0 & 0\\\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

  then the 1'st, the 3'rd, and the 4'th columns contain a leading 1 and thus $col_1(A), col_3(A), col_4(A)$ form a basis of the column space of A.

Note:

To find the basis of the column space is to find to basis for the vector space $span\{col_1(A), col_2(A), \ldots, col_n(A)\}$. Two methods introduced in the previous section can also be used. The method used in this section is equivalent to the second method in the previous section.

Example:

Let

$$A=\begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}.$$

Find the bases of the row and column spaces of A.

[solution:]

Step 1:

Transform the matrix A to the matrix in reduced row echelon form,

$$A=\begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix} \xrightarrow{\text{in reduced row echelon form}} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 2:

- The basis for the row space is

$$\left\{\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & -1 \end{bmatrix}\right\}$$

- The columns corresponding to the ones containing the leading 1's are the 1'st, the 2'nd, and the 4'th columns. Thus,

$$\left\{\begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}\right\}$$

form a basis of the column space.

Definition of row rank and column rank:

The dimension of the row space of A is called the row rank of A and the dimension of the column space of A is called the column rank of A.

Example

Since the basis of the row space of A is

$$\left\{\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & -1 \end{bmatrix}\right\},$$

the dimension of the row space is 3 and the row rank of A is 3. Similarly,

$$\left\{\begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \\ 4 \end{bmatrix}\right\}$$

is the basis of the column space of A. Thus, the dimension of the column space is 3 and the column rank of A is 3.

Important Result:

The row rank and column rank of the $m \times n$ matrix A are equal.

Definition of the rank of a matrix:

Since the row rank and the column rank of a $m \times n$ matrix A are equal, we only refer to the rank of A and write $rank(A)$.

Important Result:

If A is a $m \times n$ matrix , then

$$\begin{aligned} & rank(A) + nullity(A) \\ & = \text{the dimension of the column space} + \text{the dimension of the null space} \\ & = n \end{aligned}$$

Example $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $n = 5$.

Since

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

is a basis of column space and thus $rank(A) = 3$. The solutions of $Ax = 0$ are

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = s_1, x_5 = s_2,\ s_1, s_2 \in R.$$

Thus, the solution space (the null space) is

$$s_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow\ span\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Then, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ are the basis of the null space. and $nullity(A) = 2$.

Therefore,

$$rank(A) + nullity(A) = 3 + 2 = 5 = n.$$

Important Result:

Let A be an $n \times n$ matrix.

A is nonsingular if and only if $rank(A) = n$.

$$rank(A) = n \iff A \text{ is nonsingular} \iff \det(A) \neq 0$$
$$\iff Ax = b \text{ has a unique solution.}$$
$$rank(A) < n \iff Ax = 0 \text{ has a nontrivial solution.}$$

Important Result:
Let A be an $m \times n$ matrix. Then,
$Ax = b$ has a solution $rank(A) = rank[A \mid b]$

Eigenvectors and Eigenvalues of a matrix

The eigenvectors of a square matrix are the non-zero vectors which, after being multiplied by the matrix, remain proportional to the original vector, i.e. any vector $\mathbf{x}$ that satisfies the equation:

$$\mathbf{Ax} = \lambda\mathbf{x},$$

where $\mathbf{A}$ is the matrix in question, $\mathbf{x}$ is the eigenvector and λ is the associated eigenvalue.

As will become clear later on, eigenvectors are not unique in the sense that any eigenvector can be multiplied by a constant to form another eigenvector. For each eigenvector there is only one associated eigenvalue, however.

If you consider a 2×2 matrix as a stretching, shearing or reflection transformation of the plane, you can see that the eigenvalues are the lines passing through the origin that are left unchanged by the transformation[1].

Note that square matrices of any size, not just 2×2 matrices, can have eigenvectors and eigenvalues.

In order to find the eigenvectors of a matrix we must start by finding the *eigenvalues*. To do this we take everything over to the LHS of the equation:

$$\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0},$$

then we pull the vector $\mathbf{x}$ outside of a set of brackets:

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}.$$

The only way this can be solved is if $\mathbf{A} - \lambda\mathbf{I}$ does not have an inverse[2], therefore we find values of λ such that the determinant of $\mathbf{A} - \lambda\mathbf{I}$ is zero:

---

[1] We leave out rotations for the moment as no vector other than the zero vector (the origin) is left unchanged. We will see later there is a way of coping with rotation.
If $\mathbf{A} - \lambda\mathbf{I}$ does have an inverse we find $\mathbf{x} = (\mathbf{A} - \lambda\mathbf{I})^{-1}\mathbf{0} = \mathbf{0}$, i.e. the only solution is the zero vector.

$$|\mathbf{A} - \lambda \mathbf{I}| = 0.$$

Once we have a set of eigenvalues we can substitute them back into the original equation to find the eigenvectors. As always, the procedure becomes clearer when we try some examples:

Example 1

Q) Find the eigenvalues and eigenvectors of the matrix:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

A) First we start by finding the eigenvalues, using the equation derived above:

$$|\mathbf{A} - \lambda \mathbf{I}| = \left| \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}.$$

If you like, just consider this step as, “subtract λ from each diagonal element of the matrix in the question”.

Next we derive a formula for the determinant, which must equal zero:

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - 1 \times 1 = \lambda^2 - 2\lambda + 3 = 0.$$

We now need to find the roots of this quadratic equation in λ.

In this case the quadratic factorises straightforwardly to:

$$\lambda^2 - 2\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0.$$

The solutions to this equation are $\lambda_1 = 1$ and $\lambda_2 = 3$. These are the eigenvalues of the matrix $\mathbf{A}$.

We will now solve for an eigenvector corresponding to each eigenvalue in turn. First we will solve for $\lambda = \lambda_1 = 1$:

To find the eigenvector we substitute a general vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ into the defining equation:

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x},$$
$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1 \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

By multiplying out both sides of this equation, we form a set of simultaneous equations:

$$\begin{pmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

or

$$\begin{aligned} 2x_1 + x_2 &= x_1, \\ x_1 + 2x_2 &= x_2. \end{aligned}$$

$$\begin{aligned} x_1 + x_2 &= 0, \\ x_1 + x_2 &= 0, \end{aligned}$$

where we have taken everything over to the LHS. It should be immediately clear that we have a problem as it would appear that these equations are not solvable! However, as we have already mentioned, the eigenvectors are not unique: we would not expect to be able to solve these equation for one value of x_1 and one value of x_2. In fact, all these equations let us do is specify a relationship between x_1 and x_2, in this case:

$$x_1 + x_2 = 0,$$

or,

$$x_2 = -x_1,$$

so our eigenvector is produced by substituting this relationship into the general vector $\mathbf{x}$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix}.$$

This is a valid answer to the question, however it is common practice to put 1 in place of x_1 and give the answer:

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We follow the same procedure again for the second eigenvalue, $\lambda = \lambda_2 = 3$. First we write out the defining equation:

$$\mathbf{Ax} = \lambda \mathbf{x},$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3 \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and multiply out to find a set of simultaneous equations:

$$\begin{aligned} 2x_1 + x_2 &= 3x_1, \\ x_1 + 2x_2 &= 3x_2. \end{aligned}$$

Taking everything over to the LHS we find:

$$-x_1 + x_2 = 0,$$
$$x_1 - x_2 = 0.$$

This time both equations can be made to be the same by multiplying one of them by minus one. This is used as a check: one equation should always be a simple multiple of the other; if they are not and can be solved uniquely then you have made a mistake!

Once again we can find a relationship between x_1 and x_2, in this case $x_1 = x_2$, and form our general eigenvector:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}.$$

As before, set $x_1 = 1$ to give:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore our full solution is:

$$\lambda_1 = 1, \quad \mathbf{x}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix};$$

$$\lambda_2 = 3, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Example 2

Q) You will often be asked to find *normalised* eigenvectors. A normalised eigenvector is an eigenvector of length one. They are computed in the same way but at the end we divide by the length of the vector found. To illustrate, let's find the normalised eigenvectors and eigenvalues of the matrix:

$$\mathbf{A} = \begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix}.$$

A) First we start by finding the eigenvalues using the eigenvalues equation:

$$|\mathbf{A} - \lambda\mathbf{I}| = \left|\begin{pmatrix} 5-\lambda & -2 \\ 7 & -4-\lambda \end{pmatrix}\right| = \mathbf{0}.$$

Computing the determinant, we find:

$$(5-\lambda)(-4-\lambda)+2\times 7=0,$$

and multiplying out:

$$\lambda^2-\lambda-6=0.$$

This quadratic can be factorised into $(\lambda-3)(\lambda+2)=0$, giving roots $\lambda_1=-2$ and $\lambda_2=3$.

To find the eigenvector corresponding to $\lambda=\lambda_1=-2$ we must solve:

$$\mathbf{Ax}=\lambda\mathbf{x},$$
$$\begin{pmatrix}5 & -2\\ 7 & -4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=-2\times\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

When we compute this matrix multiplication we obtain the two equations:

$$5x_1-2x_2=-2x_1,$$
$$7x_1-4x_2=-2x_2.$$

Moving everything to the LHS we once again find that the two equations are identical:

$$7x_1-2x_2=0,$$
$$7x_1-2x_2=0,$$

and we can form the relationship $x_2=\frac{7}{2}x_1$ and the eigenvector in this case is thus:

$$\mathbf{x}=\begin{pmatrix}x_1\\ \frac{7}{2}x_1\end{pmatrix}.$$

In previous questions we have set $x_1=1$, but we were free to choose any number. In this case things are made simpler by electing to use $x_1=2$ as this gets rid of the fraction, giving:

$$\mathbf{x}=\begin{pmatrix}2\\ 7\end{pmatrix}.$$

This is not the bottom line answer to this question as we were asked for normalized eigenvectors. The easiest way to normalize the eigenvector is to divide by its length, the length of this vector is:

$$|\mathbf{x}|=\sqrt{2^2+7^2}=\sqrt{53}.$$

Therefore the normalized eigenvector is:

$$\hat{\mathbf{x}} = \frac{1}{\sqrt{53}}\begin{pmatrix} 2 \\ 7 \end{pmatrix},$$

The chevron above the vector's name denotes it as normalised. It's a good idea to confirm that this vector does have length one:

$$|\hat{\mathbf{x}}| = \sqrt{\left(\frac{2}{\sqrt{53}}\right)^2 + \left(\frac{7}{\sqrt{53}}\right)^2} = \sqrt{\frac{4}{53} + \frac{49}{53}} = \sqrt{\frac{53}{53}} = 1.$$

We must now repeat the procedure for the eigenvalue $\lambda = \lambda_2 = 3$. We find the simultaneous equations are:

$$\begin{aligned} 2x_1 - 2x_2 &= 0, \\ 7x_1 - 7x_2 &= 0, \end{aligned}$$

and note that they differ by a constant ratio. We find the relation between the components, $x_1 = x_2$, and hence the general eigenvector:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix},$$

and choose the simplest option $x_1 = 1$ giving:

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This vector has length $\sqrt{1+1} = \sqrt{2}$, so the normalised eigenvector is:

$$\hat{\mathbf{x}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore the solution to the problem is:

$$\lambda_1 = -2, \quad \hat{\mathbf{x}}_1 = \frac{1}{\sqrt{53}}\begin{pmatrix} 2 \\ 7 \end{pmatrix};$$
$$\lambda_2 = 3, \quad \hat{\mathbf{x}}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Example 3

Q) Sometimes you will find complex values of λ; this will happen when dealing with a rotation matrix such as:

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which represents a rotation though $90°$. In this example we will compute the eigenvalues and eigenvectors of this matrix.

A) First start with the eigenvalue formula:

$$|\mathbf{A}-\lambda\mathbf{I}| = \left|\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}\right| = \mathbf{0}.$$

Computing the determinant we find:

$$\lambda^2 + 1 = 0,$$

which has complex roots $\lambda = \pm i$. This will lead to complex-valued eigenvectors, although there is otherwise no change to the normal procedure.

For $\lambda_1 = i$ we find the defining equation to be:

$$\mathbf{Ax} = \lambda\mathbf{x},$$
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = i \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Multiplying this out to give a set of simultaneous equations we find:

$$-x_2 = ix_1,$$
$$x_1 = ix_2.$$

We can apply our check by observing that these two equations can be made the same by multiplying either one of them by i. This leads to the eigenvector:

$$\mathbf{x} = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Repeating this procedure for $\lambda = \lambda_2 = -i$, we find:

$$\mathbf{x} = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Therefore our full solution is:

$$\lambda_1 = i, \quad \mathbf{x}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix};$$
$$\lambda_2 = -i, \quad \mathbf{x}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

UNIT - II
LIMITS & CONTINUITY

Contents

LIMITS & CONTINUITY:
Limit at a Point, Properties of Limit, Computation of Limits of Various Types of Functions, Continuity at a Point, Continuity Over an Interval, Intermediate Value Theorem, Type of Discontinuities

Limit – **used to describe the way a function varies**.

a) some vary continuously – small changes in x produce small changes in f(x)
b) others vary erratically or jump
c) is fundamental to finding the tangent to a curve or the velocity of an object

Average Speed during an interval of time = distance covered/the time elapsed

(measured in units such as: km/h, ft/sec, etc.)

Δdistance/Δtime

a) **free fall** = (discovered by Galileo) a solid object dropped from rest (not moving) to fall freely near the surface of the earth will fall a distance proportional to the square of the time it has been falling

$y = 16t^2$ y is the distance fallen after t seconds, 16 is constant of proportionality

ex. A rock breaks loose from a cliff, What is the average speed

a) during first 4 seconds of fall
b) during the 1 second interval between 2 sec. And 3 sec.

a) $\frac{\Delta y}{\Delta t} \quad \frac{16(4)^2 - 16(0)^2}{4 - 0} \quad \frac{256}{4} \quad 64\text{ft/sec}$

c) $\frac{16(3)^2 - 16(2)^2}{3 - 2} \quad 80 \text{ ft/sec}$

Average Rates of Change and Secant Lines: find by dividing the change in y by the length of the interval:

Average rate of change of y = f(x) with respect to x over interval $[x_1,x_2]$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h} \qquad h \neq 0$$

Geometrically the rate of change of f over the above interval is the slope of the line through two point of the function(curve) = **Secant

Example 3 and Example 4 p. 75 of book

LIMITS: Let f(x) be defined on an open interval about c, except possibly at c itself

** if f(x) gets very close to L, for all x sufficiently close to c we say that f Approaches the limit L written as:

$\lim_{x \to c} f(x) = L$ "the limit of f(x) approaches c = L (in book call c x_0)

** underlying idea of limit – behavior of function near x=c rather than at x = c

** when approaching from left and right – must approach same #, not Different or else no limit exists

Ex. suppose you want to describe the behavior of: when x is very close to 4

$$f(x) = \frac{.1x^4 - .8x^3 + 1.6x^2 + 2x - 8}{x - 4}$$

a) first of all the function is not defined when x = 4

b) to see what happens to the values of f(x) when x is very close to 4, observe the graph of the function in the viewing window $3.5 \le x \le 4.5$ and $0 \le y \le 3$ -- use the trace feature to move along the graph and examine The values of f(x) as x gets closer to 4 (can use table function on calc)

c) also notice the "hole" at 4

d) the exploration and table show that as x gets closer to 4 from either side (+/-) the corresponding values of f(x) get closer and closer to 2.

Therefore, the limit as x approaches 4 = 2 $\lim_{x \to 4} f(x) = 2$

Identity Function of Limits: for every real number c, $\lim_{x \to c} x = c$

Ex. $\lim_{x \to 2} x = 2$

Limit of a Constant: if d is a constant then $\lim_{x \to c} d = d$

Ex. $\lim_{x \to 3} 3 = 3$ $\lim_{x \to 15} 4 = 4$

Nonexistence of Limits (limit of f(x) as x approaches c may fail to exist if:

#1. f(x) becomes infinitely large or infinitely small as x approaches c from either side

Ex. $\lim_{x \to 0} \frac{1}{x^2}$

* graph in calculator – as x approaches 0 from the left or right the corresponding values of f(x) become larger and larger without bound – rather than approaching 1 particular number – therefore the limit doesn't exist!!

#2. f(x) approaches L as x approaches c from the right and f(x) approaches M with $M \neq L$, as x approaches c from the left.

Ex. $\lim_{x \to 0} \frac{|x|}{x}$

*function is not defined when x=0. and according to def. of absolute value, $|x| = x$ when $x>0$ and $|x| = -x$ when $x<0$ so 2 possibilities: if $x>0$ then $f(x) = 1$

If $x<0$ then $f(x) = -1$

* if x approaches 0 from the right,(through + values) then corresponding values always are 1

*if x approaches 0 from the left (-values) then correspond. values are always -1

** so don't approach the same real # as required by def. of limit – Therefore, the limit doesn't exist

#3. f(x) oscillates infinitely many times between numbers as x approaches c from either Side.

Ex. $\lim_{x \to 0} \sin \frac{\pi}{x}$

**If graph in calc. – see that the values oscillate between -1 and 1 infinitely many time, not approaching one particular real number – so limit doesn't exist.

Calculating using the Limit Laws:

If L,M,c and k are real numbers and:

$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ then

#1. **Sum Rule**: $\lim_{x \to c} (f+g)(x) = \lim_{x \to c}[f(x) + g(x)] = L + M$

#2. **Difference Rule**: $\lim_{x \to c}(f-g)(x) = \lim_{x \to c}[f(x) - g(x)] = L-M$

#3. **Product Rule**: $\lim_{x \to c}(f\cdot g)(x) = \lim_{x \to c}[f(x) \cdot g(x)] = L\cdot M$

#4. **Quotient Rule**: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ $M \neq 0$

#5. **Constant Multiple Rule**: $\lim_{x \to c}(k\cdot f(x)) = k\cdot L$

the limit of a constant times a function is the constant times the limit

#6. **Power Rule**: if r and s are integers with no common factors and s≠0 then:

$$\lim_{x \to c} \sqrt{f(x)} = \sqrt{L}$$

** Limits of Polynomial/Rational Functions can be found by substitution:

- If f(x) is a polynomial function and c is any real #, then

$$\lim_{x \to c} f(x) = f(c)$$

plug in c in the function

ex. $\lim_{x \to -2} (x^2+3x-6) = \lim_{x \to -2} x^2 + \lim_{x \to -2} 3x - \lim_{x \to -2} 6$ (sum and difference rule)

$\lim x \cdot \lim x + \lim 3 \cdot \lim x - \lim 6$ (product rule)

$\lim x \cdot \lim x + 3 \lim x - 6$ (limit of a constant rule)

$(-2)(-2) + 3(-2) - 6$ (limit of x/Identity rule)

$= -8$

Ex. $\lim_{x \to 2} \frac{x^3 - 3x^2 + 10}{x^2 - 6x + 1}$ (done in 1 step) $\frac{2^3 - 3(2)^2 + 10}{2^2 - 6(2) + 1}$ $\frac{6}{-7} = -.857$

** Substitution in a Rational Function works only if the denominator is not zero at the limit point c. – if it is: cancelling common factors in the numerator and denominator may create s simplified fraction where substitution may be possible:

Ex. $\lim_{x \to 3} = \frac{x^2 - 2x - 3}{x - 3}$ ** denom. Is 0 at x=3, so try to simplify

$= \frac{(x-3)(x+1)}{x - 3}$ ** cancel out new fraction = x +1

$= (3) + 1 = 4$ **can substitute now bc won't be 0 at 3

** creating a common factor so can substitute

Ex. $\lim_{x \to -1} \frac{\sqrt{x^2+8} - 3}{x+1}$

$\lim_{x \to -1} \frac{(\sqrt{x^2+8} - 3)}{x + 1} \frac{(\sqrt{x^2 + 8} + 3)}{(\sqrt{x^2 + 8} + 3)}$

$= \frac{(x^2+8) - 9}{(x+1)(\sqrt{x^2+8} + 3}$

$= \frac{(x+1)(x-1)}{(x+1)(\sqrt{x^2+8} + 3}$

$= \underline{x-1}$

$(\sqrt{x^2+8}\ +3$ (now can substitute -1)

$= -1/3$

Sandwich Theorem: refers to a function f whose values are sandwiched between the values of 2 other functions g and h that have the same limit, L, the values of f must also approach L:

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at x =c itself. Suppose also that:

$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$ then $\lim_{x \to c} f(x) = L$

Ex. if $\sqrt{5} - 2x^2 \leq f(x) \leq \sqrt{5} - x^2$ for $-1 \leq x \leq 1$ find $\lim_{x \to 0} f(x)$

$\sqrt{5} - 2(0)^2 \leq f(x) \leq \sqrt{5} - (0)^2$ $\sqrt{5} \leq f(x) \leq \sqrt{5}$

Theorem 5:

If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at x = c, itself, and the limits of f and g both exist as x approach c, then:

$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$

The Precise Definition of a Limit

Let f(x) be defined on an open interval about (c), except possibly at (c) itself. We say that the limit of f(x) as x approaches (c) is the number L and write:

$\lim_{x \to c} f(x) = L$

if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

$\mathbf{0 < |x - c| < \delta}$ **and** $\mathbf{|f(x) - L| < \varepsilon}$

*ε = indicates how close f(x) should be to the limit (the error tolerance)

*δ = indicates how close the c must be to get the L (distance from c)

Using the Definition Example:

Ex. Prove that the $\lim_{x \to 1} (2x + 7) = 9$

Steps: 1. c = 1, and L =9 so $0<|x - 1|<\delta$ and $|(2x+7) - 9|<\varepsilon$

Step 2: in order to get some idea which δ might have this property work backwards from the desired conclusion:

$|(2x+7)-9|<\varepsilon$

$|2x - 2|<\varepsilon$

$|2(x-1)|<\varepsilon$ (factor out common)

$|2|\ |x-1|<\varepsilon$

$2|x-1|<\varepsilon$ (divide by 2)

$= |x-1|<\varepsilon/2$ -- this says that $\varepsilon/2$ would be a good choice for δ

Step 3: go forward:

$|x-1|<\varepsilon/2$ (get rid of 2 by multiplying on both sides)

$2|x-1|<\varepsilon$

$|2||x-1|<\varepsilon$

$|2(x-1)|<\varepsilon$

$|2x-2|<\varepsilon$ (rewrite -2 as 7-9)

$|(2x+7)-9|<\varepsilon$

$|f(x) - 9|<\varepsilon$ therefore: $\varepsilon/2$ has required property and proven

<u>Finding δ algebraically for given epsilons</u>

The process of finding a $\delta>0$ such that for all x:

$0<|x – c|<\delta$ ----- $|f(x) - L|<\varepsilon$ can be accomplished in 2 ways:

1. Solve the inequality $|f(x) - L|<\varepsilon$ to find an open interval (a,b) containing x_0 on Which the inequality holds for all $x\neq c$

2. Find a value of $\delta>0$ that places the open interval $(c – \delta, c + \delta)$ centered at x_0 inside the interval (a,b). The inequality $|f(x)-L|<\varepsilon$ will hold for all $x\neq c$ in This δ-interval

Ex. Find a value of $\delta>0$ such that for all x, $0<|x-c|<\delta$ ---- $a<x<b$

If a=1 b=7 c=2 so $1<x<7$

Step 1: $|x-2|<\delta$ --- $-\delta<x-2<\delta$ --- $-\delta+2<x<\delta+2$

Step 2: a) $-\delta+2 =1$ $-\delta=-1$ --- $\delta = 1$

b) $\delta+2 = 7$ $\delta=5$ **closer to a endpoint

therefore: the value of δ which assures $|x-2|<\delta$ $1<x<7$ is smaller value $\delta=1$

Ex. Find an open interval about c on which the inequality $|f(x) - L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-c|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

If $f(x)=\sqrt{x}$ $L=½$ $c=¼$ $\varepsilon=0.1$

Step 1: $|\sqrt{x} -½|<0.1$ --- $-0.1<\sqrt{x} - ½<0.1$ --- $0.4<\sqrt{x}<.6$ --- $0.16<x<.36$

Step 2: $0<|x-¼|<\delta$ --- $-\delta<x - ¼<\delta$ --- $-\delta+¼<x<\delta+¼$

a) $-\delta+¼ =.16$ -- $-\delta.=-09$ -- $\delta=.09$

b) $\delta+¼=.36$ --- $\delta= .11$

Therefore, $\delta=.09$

Ex. With the given f(x), point c and a positive number ε, Find $L = \lim_{x \to x_0} f(x)$

then find a number δ>0 such that for all x

$f(x)=-3x-2$ $x_0=-1$ $\varepsilon=.03$ $\lim (-3x-2) = (-3)(-1)-2 = 1$

Step 1: $|f(x)-L|<\varepsilon = |(-3x-2)-1|<.03 = -.03<-3x-3<.03 = -1.01<x<-.99$

Step 2: $|x-x_0|<\delta = |x-(-1)|<\delta = -\delta<x+1<\delta = -\delta-1<x<\delta-1$

a) $-\delta-1 =-1.01$ distance to nearer endpoint of -1.01 = .01

b) $\delta-1=-.99$ distance to nearer endpoint of -.99 =.01 therefore: $\delta=.01$

Two Sided Limits – what we dealt with in section 1, as x approaches c, a function,f, Must be defined on both sides of c and its values f(x) must approach L as x approaches c from either side.

One-Sided Limit – a limit if the approach is only from one side.

a) **Right-hand limit** = if the approach is from the right

$\lim_{x \to c^+} f(x) = L$

where x>c

b) **Left-hand limit** = if the approach is from the left

$\lim_{x \to c^-} f(x) = L$

where x<c

** All properties listed for two sided limits apply for one side limits also.

Two Sided Limit Theorem; a function f(x) has a limit as x approaches c if and only if it has left-handed and right hand limits there and the one sided limits equal:

$\lim_{x \to c} f(x) = L$ if and only if: $\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$

Precise Definitions of Right Hand and Left Hand Limits:

F(x) has right hand limit at x_0(c) and write:

$\lim_{x \to x_0} f(x) =L$ if for every number ε>0 there exists a corresponding number δ>0 such that for all x $x_0<x<x_0+\delta$ ---- $|f(x) - L|<\varepsilon$

f(x) has left hand limit at x_0(c) and write

$\lim_{x \to x_0} f(x) = L$ if for every number ε>0 there exists a corresponding number δ>0 such that for all x $x_0-\delta<x<x_0$ ---- $|f(x) – L|<\varepsilon$

<u>Theorem 7 involving Sin</u>. – in radian measure its limit as $\Theta \to 0 = 1$ so…

$$\lim_{\Theta \to 0} = \frac{\sin\Theta}{\Theta} = 1 \qquad (\Theta \text{ in radians})$$

Finite Limits as x $\pm\infty$ (have outgrown their finite bounds)

<u>Definition: Limit as x approaches ∞ or -∞:</u>

1. say f(x) has the limit L as x approaches infinity and write:

   $$\lim_{x \to \infty} f(x) = L$$

   if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x: $x > M$

2. say f(x) has the limit L as x approaches minus infinity and write:

   $$\lim_{x \to -\infty} f(x) = L$$

   if for every number $\varepsilon > 0$, there exists a corresponding number N such that for all x : $x < N$

Properties of Infinite Limits:

1. $\lim_{x \to \pm\infty} k = k$ Constant function
2. $\lim_{x \to \pm\infty} \frac{1}{x} = 0$ Identity function
3. Sum, Difference, Product, Constant Multiple, Quotient, Power Rule all the same with infinity limits as with regular limits.

<u>Limits of Rational Functions</u>: -- divide the numerator and denominator by the highest power of x in the denominator.—what happens depends then on the degree of the polynomial:

a) numerator and denominator of the same degree ex. 8 p. 109
b) numerator degree less than denominator degree ex. 9 p. 109

<u>Horizontal Asymptotes</u>

A line y = b is a horizontal asymptote of the graph of a function y = f(x) if either:

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b$$

for the graph on p. 109 of the polynomial function – the as you approach 5/3 from the left and the right, the curves go to ∞ and -∞ ---the asymptote serves as like a stop sign that turns the curve towards infinity

Oblique (slanted) Asymptotes: if the degree of the numerator of a rational function is one greater than the degree of the denominator.

Infinite Limits and Vertical Asymptotes

Ex. Find the $\lim_{x \to 0^+} \frac{1}{3x} = \infty$

$\lim_{x \to 0^-} \frac{1}{3x} = -\infty$

so $\lim_{x \to 0} \frac{1}{3x}$ does not exist because the limits are not the same

Ex. Find $\lim_{x \to 7} \frac{4}{(x-7)^2}$ (check 7^- and 7^+ both are ∞, so limit exists as ∞)

Vertical Asymptote – a line x = a is a vertical asymptote of the graph of a function y= f(x) if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$

** many times a graph will have both a horizontal and vertical asymptote

Ex. Find the horizontal and vertical asymptotes of the curve:

$$Y = \frac{x + 4}{x - 3}$$

a) vertical asymptotes – look at denominator – what would make it = 0 (3) so the vertical asymptote will be at 3

b) horizontal – since first term in numerator and denominator are the same degree, look at the # in front of the terms = 1 (or view it as dividing x+2 into x+3 that will end up with a remainder of 1

Find the horizontal and vertical asymptotes of $f(x) = \frac{-8}{x^2-4}$

** The curves of y = sec x amd y = tan x have infinitely many vertical asymptotes at the odd multiples of π/2

** The curves of y = csc x and y = cot x have infinitely many vertical asymptotes at the Odd Multiples of π
(pictures on p. 119)

Rational Functions with degree of Numerator greater than degrees of denominator:

a) need to determine the horizontal asymptote by dividing numerator into denominator:

Ex. $y = \frac{x^2 - 4}{x - 1}$

Vertical Asymptote = 1 (bc mak~~es the de~~nominator = 0)

Horizontal Asymptote = $x - 1 \quad x^2 - 4$

$$= x + 1 - \frac{3}{x-1}$$

****whenever the Numerator is larger than denominator – will get an OBLIQUE ASYMPTOTE – which is a diagonal line through 1**

a) the x+1 in the horizontal asymptote dominates the asymptote when x is numerically large, and the remainder part dominates when x is numerically small. These are therefore: **Dominant Terms**

-- Continuity

Continuous – if you can draw a graph of f(x) at or a certain point without lifting your pencil.

Discontinuous – anytime there is a break, gap or hole at a point in the curve

a) point of discontinuity – the point where the gap/jump is

Right-Continuous – continuous from the right – at a point x=c in its domain if

$$\lim_{x \to c+} f(x) = f(c)$$

Left-Continuous – continuous from left- at a point c if $\lim_{x \to c-} f(x) = f(c)$

Continuity at a point:

#1 At an Interior Point – if function y = f(x) is continuous on interior point c of its domain if: $\lim_{x \to c} f(x) = f(c)$

#2. At an Endpoint – y=f(x) is continuous at a left endpoint a, or at right endpoint b, if:

$$\lim_{x \to a+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b-} f(x) = f(b)$$

Ex. Without graphing, show that the function $f(x) = \frac{\sqrt{2x}\,(2-x)}{x^2}$ is continuous at x = 3

step 1: show $f(3) = \frac{\sqrt{2x}(2-x)}{x^2} = \frac{\sqrt{2(3)}\cdot(2-3)}{3^2} = \frac{\sqrt{6}}{-9}$

step 2: show $\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{\sqrt{2x}\,(2-x)}{x^2}$ = limit of quotient $\frac{\lim \sqrt{2x}\,(2-x)}{\lim x^2}$

$= \frac{\lim \sqrt{2x} \cdot \lim(2-x)}{\lim x^2}$ limit of a product

$= \frac{\sqrt{\lim 2x} \cdot \lim(2-x)}{\lim x^2}$ = limit of a root

$$= \frac{\sqrt{6} \cdot (-1)}{9} \qquad = \frac{\sqrt{6}}{9}$$

** so lim f(x) = f(3) and is continuous at x = 3

Definition of Continuity/Continuity Test:

A function f(x) is continuous at x = c if and only if it meets the following 3 conditions:

1. f(c) exists – c lies in the domain of f
2. $\lim_{x \to c} f(x)$ exists (f has a limit as x approaches c)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

Continuity of Special Functions:

a) Every polynomial function is continuous at every real #
b) Every rational function is continuous at every real # in its domain
c) Every exponential function is continuous at every real #
d) Every logarithmic function is continuous at every positive real #
e) F(x) = sin x and g(x) = cos x are continuous at every real #
f) H(x) = tan x is continuous at every real # in its domain

Continuity on the Interval: - a function is continuous on the interval if and only if it is continuous at every point of the interval.

- a function is continuous on the closed interval [a,b] provided that f is continuous from the right at x= a and from the left at x=b and continuous at every value in the open int. (a,b)

Properties of Continuous functions:

If the functions f and g are continuous at x=c, then the following combinations are continuous at x = c

1. Sums: f + g
2. Differences: f-g
3. Products: f·g
4. Constant Multiples: k·f for any # k
5. Quotients: f/g provided $g(c) \neq 0$
6. Powers: $f^{r/s}$ provided it is defined on the open interval containing c, and r,s are integer

Continuity of Composite Functions: the function f is continuous at x=c and the function g is continuous at x = f(c), then the composite function g◦f is continuous at x = c.

Ex. Show that $h(x) = \sqrt{x^3 - 3x^2 + x + 7}$ is continuous at x = 2

Steps: first show $f(2) = 2^3 - 3(2)^2 + 2 + 7 = 5$

Then check $g(x) = \sqrt{x}$ which is continuous b/c by power property $\sqrt{\lim_{x \to 5} x} = \sqrt{5}$

So, with c=2 and f(c)=5, the composite function g◦f given by:

$(g \circ f)(x) = (g(f(x)) = g(x^3 - 3x^2 + x + 7) = \sqrt{x^3 - 3x^2 + x + 7}$

Ex. $x^{2/3}$

$1+x^4$ is this continuous everywhere on their respective domains

Yes, because the numerator if a rational power of the identity function, and the Denominator is an everywhere positive polynomial

Continuous Extension to a Point – often a functions (such as a rational function) may have a limit even at a point where it is not defined.

**if f(c) is not defined, but $\lim_{x \to c} f(x) = L$ exists, a new function rule can be defined as:

$$f(x) = \begin{cases} f(x) & \text{if } x \text{ is in the domain of } f \\ L & \text{if } x = c \end{cases}$$

** in rational functions, f, continuous extensions are usually found by cancelling common factors.

Ex. show that $f(x)=\dfrac{x^2+x-6}{x^2-4}$ has a continuous extension to x=2, find the extension

steps: first factor $\dfrac{(x-2)(x+3)}{(x-2)(x+2)} = \dfrac{(x+3)}{(x+2)}$ which is equal to f(x) for $x\neq 2$, but is continuous at x=2

Shows continuous by plugging in 2 to new function

$\dfrac{(2+3)}{(2+2)} = \dfrac{5}{4}$ ** have removed the point of discontinuity at 2

<u>Intermediate Value Theorem for Continuous Functions</u>

**A function y = f(x) that is continuous on a closed interval [a,b] takes on every value between f(a) and f(b). In other words, if y_0 is any value between f(a) and f(b) then $y_0 = f(c)$ for some c in [a,b]

- What this is saying Geometrically is that – any horizontal line $y=y_0$ crossing the y-axis between the numbers f(a) and f(b) will cross the curve y=f(x) at least once over the interval
- Look at figure on p.131
- For this theorem-the curve <u>must be continuous</u> with no jumps/breaks
- This theorem tells us that if f is continuous, then any interval on which f changes signs contains a **zero/ root** of the function

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Tangents and Derivatives

Geometrically speaking – what is a tangent line?

We will now study it a bit further – finding the tangent to an arbitrary curve at point $P(x_0,f(x_0))$

To do this we must:

1. calculate the slope of the secant through P and a point $Q(x_0+h, f(x_0+h))$
2. Then investigate the limit of the slope as h approaches 0
   a) if limit exists—we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope

The slope of the curve y=f(x) at the point $P(x_0,f(x_0))$ is the following:

$$m= \lim_{h \quad 0} \frac{f(x_0+h) - f(x_0)}{h} \qquad \text{(provided the limit exists)}$$

The tangent line to the curve at P is the line through P with this slope.

$$y=y_0 + m(x-x_0)$$

Difference Quotient of F: $\mathbf{\dfrac{f(x_0 + h) - f(x_0)}{h}}$

a) has a limit as h approaches 0 called the derivative of f at x_0
   1) if interpreted as the secant slope—the derivative gives the slope of the curve and tangent at the point where $x=x_0$
   2) if interpreted at the average rate of change (as in 2.1) – the derivative gives the function's rate of change with respect to x at $x=x_0$

Ex. Find an equation for the tangent to the curve at the given pt. Then sketch the curve and tangent together.

$y= (x-1)^2 +1$ at pt (1,1)

$= \lim_{x \quad 0} \dfrac{[(1+h-1)^2+1- [(1-1)^2+1]}{h} = \lim \dfrac{h^2}{h}$

$= \lim h = 0$ (b/c constant), so at (1,1) $y=1+0(x-1)$, $y=1$ is tangent line

Ex. Find the slope of the function's graph at the given pt. Then find an equation for the line tangent to the graph there.

$F(x) = x-2x^2$ (1,-1)

$\text{Lim}_{x \quad 0} \dfrac{[(1+h)-2(1+h)^2]-[1-2(1)^2]}{h} = \dfrac{(1+h-2-4h-2h^2) + 1}{h} = \lim \dfrac{h(-3-2h)}{h} = -3$

At (1,-1) = $y +1 = -3(x-1)$

Identifying Discontinuities

The three types of discontinuities are easily identified by the cartoonish graphs found in the textbook. However, hole and jump discontinuities are invisible on graphing calculators. Therefore, you must be able to identify the discontinuities algebraically.

1. Zeros in Denominators of Rational Functions: could be removable or nonremovable discontinuities.

2. Holes in Piecewise Functions: these occur when there is a singular *x*-value that is not in the domain of the function.

3. Steps in Piecewise Functions: these occur when the endpoints of adjacent branches don’t match up.

4. Toolkit Functions: you must be familiar enough with the elementary functions to be able to identify vertical asymptotes, i.e. $\tan(x)$ and $\ln(x)$.

5. Plot with a Calculator: for unfamiliar functions, you may be able to identify vertical asymptotes and steps by simply graphing the function. However, remember that holes cannot be seen on the graphs of calculators. Also, you may want to plot the functions in “dot mode” so that vertical asymptotes don’t appear to be part of the function.

6. TABLE: If you suspect that there is a discontinuity at a particular *x*-value, check the table on your calculator. If an *x*-value has an ERROR, then there is a discontinuity.

UNIT - III
DIFFERENTIATION

Contents

I. Notations for the Derivative

The derivative of $y = f(x)$ may be written in any of the following ways:

$$f'(x), \qquad y', \qquad \frac{dy}{dx}, \qquad \frac{d}{dx}\left[f(x)\right], \quad \text{or} \quad D_x\left[f(x)\right].$$

II. Basic Differentiation Rules

A. Suppose c and n are constants, and f and g are differentiable functions.

(1) $f(x) = cg(x)$

$$f'(x) = \lim_{b\to x}\frac{f(b)-f(x)}{b-x} = \lim_{b\to x}\frac{cg(b)-cg(x)}{b-x} = c\lim_{b\to x}\frac{g(b)-g(x)}{b-x} = cg'(x)$$

(2) $f(x) = g(x) \pm k(x)$

$$f'(x) = \lim_{b\to x}\frac{f(b)-f(x)}{b-x} = \lim_{b\to x}\frac{[g(b)\pm k(b)]-[g(x)\pm k(x)]}{b-x} =$$

$$\lim_{b\to x}\frac{g(b)-g(x)}{b-x} \pm \lim_{b\to x}\frac{k(b)-k(x)}{b-x} = g'(x) \pm k'(x)$$

(3) $f(x) = g(x)k(x)$

$$f'(x) = \lim_{b\to x}\frac{f(b)-f(x)}{b-x} = \lim_{b\to x}\frac{g(b)k(b)-g(x)k(x)}{b-x} =$$

$$\lim_{b\to x}\frac{g(b)k(b)-g(b)k(x)+g(b)k(x)-g(x)k(x)}{b-x} =$$

$$\left[\lim_{b\to x} g(b)\right]\left[\lim_{b\to x}\frac{k(b)-k(x)}{b-x}\right]+\left[\lim_{b\to x} k(x)\right]\left[\lim_{b\to x}\frac{g(b)-g(x)}{b-x}\right]=$$

$g(x)k'(x)+k(x)g'(x)$ (**Product Rule**)

(4) $f(x)=\dfrac{g(x)}{k(x)} \Rightarrow f(x)k(x)=g(x) \Rightarrow g'(x)=f(x)k'(x)+k(x)f'(x) \Rightarrow$

$$f'(x)=\frac{g'(x)-f(x)k'(x)}{k(x)}=\frac{g'(x)-\left[\dfrac{g(x)}{k(x)}\right]k'(x)}{k(x)}=\frac{k(x)g'(x)-g(x)k'(x)}{[k(x)]^2}.$$

This derivative rule is called the **Quotient Rule.**

(5) $f(x)=c$

$$f'(x)=\lim_{b\to x}\frac{f(b)-f(x)}{b-x}=\lim_{b\to x}\frac{c-c}{b-x}=\lim_{b\to x}\frac{0}{b-x}=\lim_{b\to x}0=0$$

(6) $f(x)=x$

$$f'(x)=\lim_{b\to x}\frac{f(b)-f(x)}{b-x}=\lim_{b\to x}\frac{b-x}{b-x}=\lim_{b\to x}1=1$$

(7) $f(x)=x^n$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}=$$

$$\lim_{h\to 0}\frac{\left[x^n+nx^{n-1}h+\dfrac{n(n-1)}{2}x^{n-2}h^2+\dots\right]-x^n}{h}=$$

$$\lim_{h\to 0}\left[\frac{nx^{n-1}h+h^2\left(\dfrac{n(n-1)}{2}x^{n-2}+\dots\right)}{h}\right]=$$

$$\lim_{h\to 0}\left[nx^{n-1}+h\left(\frac{n(n-1)}{2}x^{n-2}+\dots\right)\right]=nx^{n-1}$$ (**Power Rule)**

Example 1: Suppose f and g are differentiable functions such that $f(1)=3$,

$g(1)=7$, $f'(1)=-2$, and $g'(1)=4$. Find (i) $(f+g)'(1)$, (ii) $(g-f)'(1)$,

(iii) $(fg)'(1)$, (iv) $\left(\dfrac{g}{f}\right)'(1)$, and $\left(\dfrac{f}{g}\right)'(1)$.

$$\text{(i)}\quad (f+g)'(1) = f'(1) + g'(1) = -2 + 4 = 2$$

$$\text{(ii)}\quad (g-f)'(1) = g'(1) - f'(1) = 4 - (-2) = 6$$

$$\text{(iii)}\quad (fg)'(1) = f(1)g'(1) + g(1)f'(1) = 3(4) + 7(-2) = 12 + (-14) = -2$$

$$\text{(iv)}\quad \left(\frac{g}{f}\right)'(1) = \frac{f(1)g'(1) - g(1)f'(1)}{[f(1)]^2} = \frac{3(4) - 7(-2)}{3^2} = \frac{12+14}{9} = \frac{26}{9}$$

$$\text{(v)}\quad \left(\frac{f}{g}\right)'(1) = \frac{g(1)f'(1) - f(1)g'(1)}{[g(1)]^2} = \frac{7(-2) - 3(4)}{7^2} = \frac{-14-12}{49} = \frac{-26}{49}$$

Example 2: If $f(x) = x^4 - 3x^3 + 5x^2 - 7x + 11$, find $f'(x)$.

$$f'(x) = 4x^3 - 3(3x^2) + 5(2x) - 7(1) + 0 = 4x^3 - 9x^2 + 10x - 7$$

Example 3: If $f(x) = 4\sqrt{x} - \dfrac{3}{\sqrt[3]{x^2}} + \dfrac{5}{x} - \dfrac{7}{x^5}$, then find $f'(x)$.

$$f(x) = 4\sqrt{x} - \frac{3}{\sqrt[3]{x^2}} + \frac{5}{x} - \frac{7}{x^5} = 4x^{1/2} - 3x^{-2/3} + 5x^{-1} - 7x^{-5} \Rightarrow$$

$$f'(x) = 4\left(1/2\, x^{-1/2}\right) - 3\left(-2/3\, x^{-5/3}\right) + 5\left(-1x^{-2}\right) - 7\left(-5x^{-6}\right) =$$

$$2x^{-1/2} + 2x^{-5/3} - 5x^{-2} + 35x^{-6} = \frac{2}{\sqrt{x}} + \frac{2}{\sqrt[3]{x^5}} - \frac{5}{x^2} + \frac{35}{x^6}$$

Example 4: If $f(x) = \dfrac{x^2 + 2x - 3}{3x - 4}$, then find $f'(1)$.

$$f'(x) = \frac{(3x-4)(2x+2) - (x^2+2x-3)(3)}{(3x-4)^2} = \frac{6x^2 - 2x - 8 - 3x^2 - 6x + 9}{(3x-4)^2} =$$

3

$$\frac{3x^2 - 8x + 1}{(3x-4)^2} \Rightarrow f'(1) = \frac{3(1)^2 - 8(1) + 1}{[3(1) - 4]^2} = \frac{-4}{1} = -4 \text{ or}$$

$$f'(1) = \frac{[3(1)-4][2(1)+2] - [1^2 + 2(1) - 3](3)}{[3(1)-4]^2} = \frac{(-1)(4) - (0)(3)}{(-1)^2} = \frac{-4}{1} = -4$$

Trigonometric functions

(1) $f(x) = \sin x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} =$$

$$\lim_{h\to 0} \frac{\sin x \cosh + \cos x \sinh - \sin x}{h} = \lim_{h\to 0} \frac{\sin x(\cosh - 1) + \cos x \sinh}{h} =$$

$$(\sin x)\left[\lim_{h\to 0} \frac{\cosh - 1}{h}\right] + (\cos x)\left[\lim_{h\to 0} \frac{\sinh}{h}\right] = (\sin x)(0) + (\cos x)(1) = \quad \cos x$$

(2) $f(x) = \cos x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} =$$

$$\lim_{h\to 0} \frac{\cos x \cosh - \sin x \sinh - \cos x}{h} = \lim_{h\to 0} \frac{\cos x(\cosh - 1) - \sin x \sinh}{h} =$$

$$(\cos x)\left[\lim_{h\to 0} \frac{\cosh - 1}{h}\right] - (\sin x)\left[\lim_{h\to 0} \frac{\sinh}{h}\right] = (\cos x)(0) - (\sin x)(1) =$$

$$-\sin x$$

(3) $f(x) = \tan x = \dfrac{\sin x}{\cos x}$

$$f'(x) = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

(4) $f(x) = \sec x = \dfrac{1}{\cos x}$

$$f'(x) = \frac{(\cos x)(0) - 1(-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

(5) $f(x) = \csc x = \dfrac{1}{\sin x}$

$$f'(x) = \frac{(\sin x)(0) - 1(\cos x)}{(\sin x)^2} = \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

(6) $f(x) = \cot x = \dfrac{\cos x}{\sin x}$

$$f'(x) = \frac{(\sin x)(\sin x) - (\cos x)(\cos x)}{(\sin x)^2} = \frac{-\cos^2 x - \sin^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$$

C. Composition and the generalized derivative rules

(1) $f(x) = (g \circ k)(x) = g(k(x))$

$$f'(x) = \lim_{b \to x} \frac{f(b) - f(x)}{b - x} = \lim_{b \to x} \frac{g(k(b)) - g(k(x))}{b - x} = \lim_{b \to x} \frac{g(k(b)) - g(k(x))}{b - x} \cdot$$

$$\frac{k(b) - k(x)}{k(b) - k(x)} = \lim_{b \to x} \frac{g(k(b)) - g(k(x))}{k(b) - k(x)} \cdot \lim_{b \to x} \frac{k(b) - k(x)}{b - x} =$$

$$\lim_{k(b) \to k(x)} \frac{g(k(b)) - g(k(x))}{k(b) - k(x)} \cdot \lim_{b \to x} \frac{k(b) - k(x)}{b - x} = g'(k(x)) \cdot k'(x).$$

This derivative rule for the composition of functions is called the **Chain Rule**.

(2) Suppose that $f(x) = g(k(x))$ where $g(x) = x^n$. Then $f(x) = [k(x)]^n$.
$g(x) = x^n \Rightarrow g'(x) = nx^{n-1} \Rightarrow g'(k(x)) = n[k(x)]^{n-1}$. Thus, $f'(x) =$
$g'(k(x)) \cdot k'(x) = n[k(x)]^{n-1} \cdot k'(x)$. This derivative rule for the power of a function is called the **Generalized Power Rule**.

(3) Suppose that $f(x) = g(k(x))$ where $g(x) = \sin x$. Then $f(x) = \sin[k(x)]$.
$g(x) = \sin x \Rightarrow g'(x) = \cos x \Rightarrow g'(k(x)) = \cos[k(x)]$. Thus, $f'(x) =$
$g'(k(x)) \cdot k'(x) = \cos[k(x)] \cdot k'(x)$.

(4) Similarly, if $f(x) = \cos[k(x)]$, then $f'(x) = -\sin[k(x)] \cdot k'(x)$.

(5) If $f(x) = \tan[k(x)]$, then $f'(x) = \sec^2[k(x)] \cdot k'(x)$.

(6) If $f(x) = \sec[k(x)]$, then $f'(x) = \sec[k(x)]\tan[k(x)] \cdot k'(x)$.

(7) If $f(x) = \cot[k(x)]$, then $f'(x) = -\csc^2[k(x)] \cdot k'(x)$.

(8) If $f(x) = \csc[k(x)]$, then $f'(x) = -\csc[k(x)]\cot[k(x)] \cdot k'(x)$.

Example 1: Suppose f and g are differentiable functions such that:

$f(1) = 9$	$f(2) = -5$	$g(1) = 2$	$g(9) = 3$
$f'(1) = -2$	$f'(2) = -6$	$g'(1) = 4$	$g'(9) = 7$

Find each of the following:
(i) $(f \circ g)'(1)$;

(ii) $(g \circ f)'(1)$;

(iii) $h'(1)$ if $h(x) = \sqrt{f(x)}$;

(iv) $j'(1)$ if $j(x) = [g(x)]^5$;

(v) $l'(1)$ if $l(x) = \dfrac{3}{[f(x)]^2}$;

(vi) $s'(1)$ if $s(x) = \sin[f(x)]$; and

(vii) $m'(1)$ if $m(x) = \sec[g(x)]$.

(i) $(f \circ g)'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot g'(1) = (-6)(4) = -24$

(ii) $(g \circ f)'(1) = g'(f(1)) \cdot f'(1) = g'(9) \cdot f'(1) = 7(-2) = -14$

(iii) $h(x) = \sqrt{f(x)} = [f(x)]^{1/2} \Rightarrow h'(x) = \frac{1}{2}[f(x)]^{-1/2} \cdot f'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}} \Rightarrow$

$$h'(1) = \frac{f'(1)}{2\sqrt{f(1)}} = \frac{-2}{2\sqrt{9}} = -\frac{1}{3}$$

(iv) $j(x) = [g(x)]^5 \Rightarrow j'(x) = 5[g(x)]^4 \cdot g'(x) \Rightarrow j'(1) = 5[g(1)]^4 \cdot g'(1) =$

$5(2)^4(4) = 320$

(v) $l(x) = \dfrac{3}{[f(x)]^2} = 3[f(x)]^{-2} \Rightarrow l'(x) = -6[f(x)]^{-3} \cdot f'(x) \Rightarrow l'(1) =$

$$\frac{-6f'(1)}{[f(1)]^3} = \frac{-6(-2)}{9^3} = \frac{12}{729} = \frac{4}{243}$$

(vi) $s'(x) = \cos[f(x)] \cdot f'(x) \Rightarrow s'(1) = \cos[f(1)] \cdot f'(1) = \cos(9) \cdot (-2) = -2\cos 9$

(vii) $m'(x) = \sec[g(x)]\tan[g(x)] \cdot g'(x) \Rightarrow m'(1) = \sec[g(1)]\tan[g(1)] \cdot g'(1) =$

$\sec(2)\tan(2) \cdot 4 = 4\sec 2\tan 2$

Example 2: If $f(x) = \sqrt[3]{2x^4 - x^2 + 5x + 2}$, then find $f'(1)$.

$$f(x) = \sqrt[3]{2x^4 - x^2 + 5x + 2} = (2x^4 - x^2 + 5x + 2)^{1/3} \Rightarrow f'(x) =$$

$$\frac{1}{3}(2x^4 - x^2 + 5x + 2)^{-2/3}(8x^3 - 2x + 5) = \frac{8x^3 - 2x + 5}{3\sqrt[3]{(2x^4 - x^2 + 5x + 2)^2}} \Rightarrow$$

$$f'(1) = \frac{8 - 2 + 5}{3\sqrt[3]{(2 - 1 + 5 + 2)^2}} = \frac{11}{3\sqrt[3]{64}} = \frac{11}{12}$$

Example 3: If $g(x) = \dfrac{4}{(x^3 + 4)^8}$, then find $g'(x)$.

$$g(x) = \frac{4}{(x^3 + 4)^8} = 4(x^3 + 4)^{-8} \Rightarrow g'(x) = -32(x^3 + 4)^{-9}(3x^2) = \frac{-96x^2}{(x^3 + 4)^9}$$

Example 4: If $h(x) = \sin(\cos x)$, then find $h'(x)$.

$$h'(x) = \cos(\cos x) \cdot (-\sin x)$$

Example 5: If $j(x) = \tan(2x^2 - 3x + 1)$, then find $j'(x)$.

$$j'(x) = \sec^2(2x^2 - 3x + 1) \cdot (4x - 3)$$

Example 6: If $k(x) = x^2\sqrt{3x+4}$, then find $k'(x)$.

$$k(x) = x^2\sqrt{3x+4} = x^2(3x+4)^{1/2} \Rightarrow k'(x) = x^2\left[\tfrac{1}{2}(3x+4)^{-1/2}(3)\right] +$$
$$(3x+4)^{1/2}(2x) = \frac{3x^2}{2(3x+4)^{1/2}} + \frac{2x(3x+4)^{1/2}}{1} = \frac{3x^2 + 4x(3x+4)}{2(3x+4)^{1/2}} =$$
$$\frac{15x^2 + 16x}{2(3x+4)^{1/2}}$$

Example 7: If $l(x) = \left(\dfrac{2x-1}{3x+4}\right)^4$, then find $l'(x)$.

$$l'(x) = 4\left(\frac{2x-1}{3x+4}\right)^3\left[\frac{(3x+4)(2) - (2x-1)(3)}{(3x+4)^2}\right] = \frac{4(2x-1)^3}{(3x+4)^3}\left[\frac{11}{(3x+4)^2}\right] =$$
$$\frac{44(2x-1)^3}{(3x+4)^5}.$$

Example 8: If $k(x) = \dfrac{\sin x}{1+\cos x}$, then find $k'(x)$.

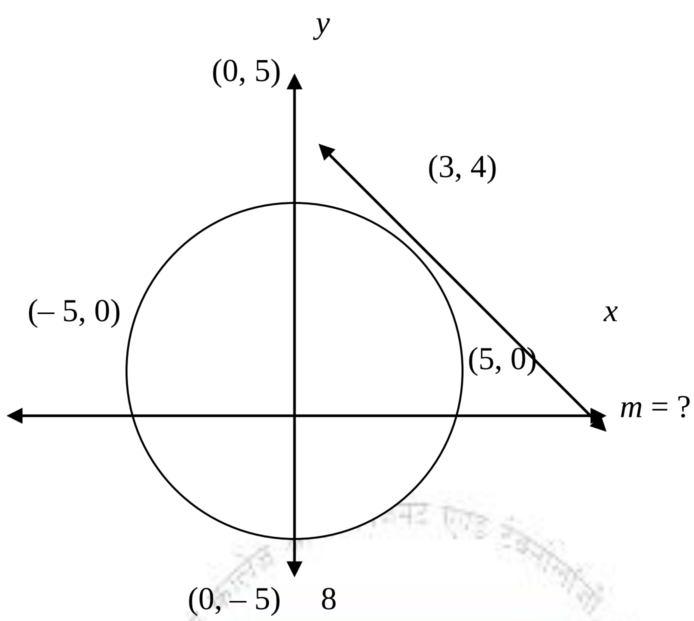
$$k'(x) = \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} =$$
$$\frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}.$$

Example 9: If $s(x) = \sin^3(x^2 - 1)$, then find $s'(x)$.

$$s(x) = \sin^3(x^2-1) = [\sin(x^2-1)]^3 \Rightarrow s'(x) = 3[\sin(x^2-1)]^2 \cdot \cos(x^2-1) \cdot 2x =$$
$$6x\sin^2(x^2-1)\cos(x^2-1).$$

Implicit Differentiation

Example 1: Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point (3, 4).



Solution 1 : A circle is not a function. However, $x^2 + y^2 = 25 \Rightarrow y^2 = 25 - x^2 \Rightarrow y = \pm\sqrt{25 - x^2} \Rightarrow y = \sqrt{25 - x^2}$ is the equation of the upper half circle and $y = -\sqrt{25 - x^2}$ is the equation of the lower half circle.

Since the point (3, 4) is on the upper half circle, use the function $f(x) = \sqrt{25 - x^2} = \left(25 - x^2\right)^{1/2} \Rightarrow f'(x) = \frac{1}{2}\left(25 - x^2\right)^{-1/2}(-2x) = \dfrac{-x}{\sqrt{25 - x^2}} \Rightarrow$

$$m = f'(3) = \frac{-3}{\sqrt{25 - 3^3}} = \frac{-3}{\sqrt{25 - 9}} = \frac{-3}{\sqrt{16}} = -\frac{3}{4}.$$

Sometimes, an equation [$x^2 + y^2 = 25$] in two variables, say x and y, is given, but it is not in the form of $y = f(x)$. In this case, for each value of one of the variables, one or more values of the other variable may exist. Thus, such an equation may describe one or more functions [$y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$]. Any function defined in this manner is said to be defined <u>implicitly</u>. For such equations, we may not be able to solve for y explicitly in terms of x [in the example, I was able to solve for y explicitly in terms of x]. In fact, there are applications where it is not essential to obtain a formula for y in terms of x. Instead, the value of the derivative at certain points must be obtained. It is possible to accomplish this goal by using a technique called **implicit differentiation**. Suppose an equation in two variables, say x and y, is given and we are told that this equation defines a differentiable function f with $y = f(x)$. Use the following steps to differentiate implicitly:

(1) Simplify the equation if possible. That is, get rid of parentheses by multiplying using the distributive property or by redefining subtraction, and

clear fractions by multiplying every term of the equation by a common denominator for all the fractions; simplify and combine like terms.

(2) Differentiate both sides of the equation with respect to *x*. Use all the relevant differentiation rules, being careful to use the **Chain Rule** when differentiating expressions involving *y*.

(3) Solve for $\frac{dy}{dx}$.

Note: It might be helpful to substitute $f(x)$ into the equation for *y* before differentiating with respect to *x*. This will remind you when you must use the generalized forms of the **Chain Rule**. Since $f'(x) = \frac{dy}{dx}$, you differentiate with respect to *x* and substitute *y* for *f*(*x*) and $\frac{dy}{dx}$ for $f'(x)$. Then you can

solve for $\frac{dy}{dx}$.

Solution 2: $x^2 + y^2 = 25 \Rightarrow x^2 + [f(x)]^2 = 25 \Rightarrow \frac{d}{dx}\left(x^2 + [f(x)]^2 = 25\right) \Rightarrow$

$$2x + 2[f(x)]f'(x) = 0 \Rightarrow f'(x) = \frac{-2x}{2[f(x)]} \Rightarrow \frac{dy}{dx} = \frac{-x}{y} \Rightarrow \left.\frac{dy}{dx}\right|_{\substack{x=3 \\ y=4}} = -\frac{3}{4}.$$

Example 2: Suppose that the equation $\frac{2}{x} + \frac{3}{y} = x$ defines a function *f* with $y = f(x)$.

Find $\frac{dy}{dx}$ and the slope of the tangent line at the point (2, 3).

Solution 1: Solve for *y*. $xy\left(\frac{2}{x} + \frac{3}{y}\right) = xy(x) \Rightarrow 2y + 3x = x^2 y \Rightarrow y = \frac{3x}{x^2 - 2} \Rightarrow$

$$\frac{dy}{dx} = \frac{(x^2-2)(3) - 3x(2x)}{(x^2-2)^2} = \frac{-3x^2 - 6}{(x^2-2)^2} \Rightarrow \left.\frac{dy}{dx}\right|_{x=2} = \frac{-18}{4} = -\frac{9}{2}$$

Solution 2: Clear fractions $\Rightarrow 2y + 3x = x^2 y \Rightarrow \frac{d}{dx}\left(2y + 3x = x^2 y\right) \Rightarrow$

$$2\frac{dy}{dx} + 3 = x^2\frac{dy}{dx} + 2xy \Rightarrow \frac{dy}{dx} = \frac{3 - 2xy}{x^2 - 2} \Rightarrow \left.\frac{dy}{dx}\right|_{\substack{x=2 \\ y=3}} = \frac{3-12}{2} = -\frac{9}{2}$$

Solution 3: $\frac{d}{dx}\left(\frac{2}{x} + \frac{3}{y} = x\right) \Rightarrow \frac{d}{dx}\left(2x^{-1} + 3y^{-1} = x\right) \Rightarrow -2x^{-2} - 3y^{-2}\frac{dy}{dx} = 1 \Rightarrow$

$$\frac{-2}{x^2} - \frac{3}{y^2}\frac{dy}{dx} = 1 \Rightarrow -2y^2 - 3x^2\frac{dy}{dx} = x^2y^2 \Rightarrow \frac{dy}{dx} = \frac{-2y^2 - x^2y^2}{3x^2} \Rightarrow$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=2 \\ y=3}} = \frac{-18-36}{12} = \frac{-54}{12} = -\frac{9}{2}$$

Example 3: If $\cos(xy) = y$, then find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(\cos(xy) = y\right) \Rightarrow -\sin(xy)\left[x\frac{dy}{dx} + y(1)\right] = \frac{dy}{dx} \Rightarrow -x\sin(xy)\frac{dy}{dx} - y\sin(xy) =$$
$$\frac{dy}{dx} \Rightarrow -y\sin(xy) = \frac{dy}{dx}\left(1 + x\sin(xy)\right) \Rightarrow \frac{dy}{dx} = \frac{-y\sin(xy)}{1 + x\sin(xy)}$$

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IV. Higher Order Derivatives

A. Notation

(1) 1st derivative (derivative of the original function $y = f(x)$): $\frac{dy}{dx} = f'(x)$

(2) 2nd derivative (derivative of the 1st derivative): $\frac{d^2y}{dx^2} = f''(x)$

(3) 3rd derivative (derivative of the 2nd derivative): $\frac{d^3y}{dx^3} = f'''(x)$

B. Distance functions

Suppose $s(t)$ is a distance function with respect to time *t*. Then $s'(t) = v(t)$ is an instantaneous velocity (or velocity) function with respect to time *t*, and $s''(t) = v'(t) = a(t)$ is an acceleration function with respect to time *t*.

Example 1: If $f(x) = x^2 \sin x$, then find $f'(x)$ and $f''(x)$.

$f'(x) = x^2\cos x + 2x\sin x$

$f''(x) = x^2(-\sin x) + 2x\cos x + 2x\cos x + 2\sin x = -x^2\sin x + 4x\cos x + 2\sin x$

Example 2: If $g(x) = \frac{2x+3}{4x-5}$, then find $g'(x)$ and $g''(x)$.

$$g'(x) = \frac{(4x-5)(2) - (2x+3)(4)}{(4x-5)^2} = \frac{8x - 10 - 8x - 12}{(4x-5)^2} = \frac{-22}{(4x-5)^2} = -22(4x-5)^{-2}$$
$$g''(x) = 44(4x-5)^{-3}(4) = 176(4x-5)^{-3} = \frac{176}{(4x-5)^3}$$

Example 3: If $x^2 + y^2 = 25$, then find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{d}{dx}\left(x^2 + y^2 = 25\right) \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{-x}{y}\right) = \frac{y(-1) - (-x)\left(\frac{dy}{dx}\right)}{y^2} = \frac{-y + x\left(\frac{-x}{y}\right)}{y^2} = \frac{-y^2 - x^2}{y^3} =$$

$$\frac{-(x^2+y^2)}{y^3} = \frac{-25}{y^3}$$

After reading this section, you should be able to

1. *understand the basics of* ***Taylor's theorem,***
2. *write transcendental and trigonometric functions as Taylor's polynomial,*
3. *use Taylor's theorem to find the values of a function at any point, given the values of the function and all its derivatives at a particular point,*
4. *calculate errors and error bounds of approximating a function by Taylor series, and*
5. *revisit the chapter whenever Taylor's theorem is used to derive or explain numerical methods for various mathematical procedures.*

The use of Taylor series exists in so many aspects of numerical methods that it is imperative to devote a separate chapter to its review and applications. For example, you must have come across expressions such as

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad (1)$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad (2)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (3)$$

All the above expressions are actually a special case of Taylor series called the **Maclaurin series**. Why are these applications of Taylor's theorem important for numerical methods? Expressions such as given in Equations (1), (2) and (3) give you a way to find the approximate values of these functions by using the basic arithmetic operations of addition, subtraction, division, and multiplication.

Example 1

Find the value of $e^{0.25}$ using the first five terms of the Maclaurin series.

Solution

The first five terms of the Maclaurin series for e^x is

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{0.25} \approx 1 + 0.25 + \frac{0.25^2}{2!} + \frac{0.25^3}{3!} + \frac{0.25^4}{4!}$$

$$= 1.2840$$

The exact value of $e^{0.25}$ up to 5 significant digits is also 1.2840.

But the above discussion and example do not answer our question of what a Taylor series is.

Here it is, for a function $f(x)$

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \cdots \quad (4)$$

provided all derivatives of $f(x)$ exist and are continuous between x and $x+h$.

What does this mean in plain English?

As Archimedes would have said (*without the fine print*), "*Give me the value of the function at a single point, and the value of all (first, second, and so on) its derivatives, and I can give you the value of the function at any other point*".

It is very important to note that the Taylor series is not asking for the expression of the function and its derivatives, just the value of the function and its derivatives at a single point.

Now the fine print: Yes, all the derivatives have to exist and be continuous between x (the point where you are) to the point, $x+h$ where you are wanting to calculate the function at. However, if you want to calculate the function approximately by using the n^{th} order Taylor polynomial, then $1^{st}, 2^{nd}, \ldots, n^{th}$ derivatives need to exist and be continuous in the closed interval $[x, x+h]$, while the $(n+1)^{th}$ derivative needs to exist and be continuous in the open interval $(x, x+h)$.

Example 2

Take $f(x)=\sin(x)$, we all know the value of $\sin\left(\frac{\pi}{2}\right)=1$. We also know the $f'(x)=\cos(x)$ and $\cos\left(\frac{\pi}{2}\right)=0$. Similarly $f''(x)=-\sin(x)$ and $\sin\left(\frac{\pi}{2}\right)=1$. In a way, we know the value of $\sin(x)$ and all its derivatives at $x=\frac{\pi}{2}$. We do not need to use any calculators, just plain differential calculus and trigonometry would do. Can you use Taylor series and this information to find the value of $\sin(2)$?

Solution

$$x=\frac{\pi}{2}$$

$$x+h=2$$

$$h=2-x$$

$$=2-\frac{\pi}{2}$$

$$=0.42920$$

So

$$f(x+h)=f(x)+f'(x)h+f''(x)\frac{h^2}{2!}+f'''(x)\frac{h^3}{3!}+f''''(x)\frac{h^4}{4!}+\cdots$$

$$x=\frac{\pi}{2}$$

$$h=0.42920$$

$$f(x)=\sin(x),\ f\left(\frac{\pi}{2}\right)=\sin\left(\frac{\pi}{2}\right)=1$$

$$f'(x)=\cos(x),\ f'\left(\frac{\pi}{2}\right)=0$$

$$f''(x)=-\sin(x),\ f''\left(\frac{\pi}{2}\right)=-1$$

$$f'''(x)=-\cos(x),\ f'''\left(\frac{\pi}{2}\right)=0$$

$$f''''(x)=\sin(x),\ f''''\left(\frac{\pi}{2}\right)=1$$

Hence

$$f\left(\frac{\pi}{2}+h\right)=f\left(\frac{\pi}{2}\right)+f'\left(\frac{\pi}{2}\right)h+f''\left(\frac{\pi}{2}\right)\frac{h^2}{2!}+f'''\left(\frac{\pi}{2}\right)\frac{h^3}{3!}+f''''\left(\frac{\pi}{2}\right)\frac{h^4}{4!}+\cdots$$

$$f\left(\frac{\pi}{2}+0.42920\right)=1+0(0.42920)-1\frac{(0.42920)^2}{2!}+0\frac{(0.42920)^3}{3!}+1\frac{(0.42920)^4}{4!}+\cdots$$

$$=1+0-0.092106+0+0.00141393+\cdots$$

$$\cong 0.90931$$

The value of $\sin(2)$ I get from my calculator is 0.90930which is very close to the value I just obtained. Now you can get a better value by using more terms of the series. In addition, you can now use the value calculated for $\sin(2)$ coupled with the value of $\cos(2)$ (which can be calculated by Taylor series just like this example or by using the $\sin^2 x + \cos^2 x \equiv 1$ identity) to find value of $\sin(x)$ at some other point. In this way, we can find the value of $\sin(x)$ for any value from $x = 0$ to 2π and then can use the periodicity of $\sin(x)$, that is $\sin(x) = \sin(x + 2n\pi), n = 1,2,\ldots$ to calculate the value of $\sin(x)$ at any other point.

Example 3

Derive the Maclaurin series of $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Solution

In the previous example, we wrote the Taylor series for $\sin(x)$ around the point $x = \frac{\pi}{2}$. Maclaurin series is simply a Taylor series for the point $x = 0$.

$$f(x) = \sin(x),\ f(0) = 0$$
$$f'(x) = \cos(x),\ f'(0) = 1$$
$$f''(x) = -\sin(x),\ f''(0) = 0$$
$$f'''(x) = -\cos(x),\ f'''(0) = -1$$
$$f''''(x) = \sin(x),\ f''''(0) = 0$$
$$f'''''(x) = \cos(x),\ f'''''(0) = 1$$

Using the Taylor series now,

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + f''''(x)\frac{h^4}{4} + f'''''(x)\frac{h^5}{5} + \cdots$$

$$f(0+h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2!} + f'''(0)\frac{h^3}{3!} + f''''(0)\frac{h^4}{4} + f'''''(0)\frac{h^5}{5} + \cdots$$

$$f(h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2!} + f'''(0)\frac{h^3}{3!} + f''''(0)\frac{h^4}{4} + f'''''(0)\frac{h^5}{5} + \cdots$$

$$= 0 + 1(h) - 0\frac{h^2}{2!} - 1\frac{h^3}{3!} + 0\frac{h^4}{4} + 1\frac{h^5}{5} + \cdots$$

$$= h - \frac{h^3}{3!} + \frac{h^5}{5!} + \cdots$$

So

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Example 4

Find the value of $f(6)$ given that $f(4) = 125$, $f'(4) = 74$, $f''(4) = 30$, $f'''(4) = 6$ and all other higher derivatives of $f(x)$ at $x = 4$ are zero.

Solution

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + \cdots$$

$$x = 4$$

$$h = 6 - 4$$
$$= 2$$

Since fourth and higher derivatives of $f(x)$ are zero at $x = 4$.

$$f(4+2) = f(4) + f'(4)2 + f''(4)\frac{2^2}{2!} + f'''(4)\frac{2^3}{3!}$$

$$f(6) = 125 + 74(2) + 30\left(\frac{2^2}{2!}\right) + 6\left(\frac{2^3}{3!}\right)$$

$$= 125 + 148 + 60 + 8$$
$$= 341$$

Note that to find $f(6)$ exactly, we only needed the value of the function and all its derivatives at some other point, in this case, $x = 4$. We did not need the expression for the function and all its derivatives. Taylor series application would be redundant if we needed to know the expression for the function, as we could just substitute $x = 6$ in it to get the value of $f(6)$.

Actually the problem posed above was obtained from a known function $f(x) = x^3 + 3x^2 + 2x + 5$ where $f(4) = 125$, $f'(4) = 74$, $f''(4) = 30$, $f'''(4) = 6$, and all other higher derivatives are zero.

Error in Taylor Series

As you have noticed, the Taylor series has infinite terms. Only in special cases such as a finite polynomial does it have a finite number of terms. So whenever you are using a Taylor series to calculate the value of a function, it is being calculated approximately.

The Taylor polynomial of order n of a function $f(x)$ with $(n+1)$ continuous derivatives in the domain $[x, x+h]$ is given by

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + \cdots + f^{(n)}(x)\frac{h^n}{n!} + R_n(x+h)$$

where the remainder is given by

$$R_n(x+h) = \frac{(h)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

where

$$x < c < x + h$$

that is, c is some point in the domain $(x, x+h)$.

Example 5

The Taylor series for e^x at point $x = 0$ is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

a) What is the truncation (true) error in the representation of e^1 if only four terms of the series are used?

b) Use the remainder theorem to find the bounds of the truncation error.

Solution

a) If only four terms of the series are used, then

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$e^1 \approx 1+1+\frac{1^2}{2!}+\frac{1^3}{3!}$$
$$=2.66667$$

The truncation (true) error would be the unused terms of the Taylor series, which then are

$$E_t = \frac{x^4}{4!}+\frac{x^5}{5!}+\cdots$$
$$=\frac{1^4}{4!}+\frac{1^5}{5!}+\cdots$$
$$\cong 0.0516152$$

b) But is there any way to know the bounds of this error other than calculating it directly? Yes,

$$f(x+h)= f(x)+ f'(x)h+\cdots+ f^{(n)}(x)\frac{h^n}{n!}+R_n(x+h)$$

where

$$R_n(x+h)=\frac{(h)^{n+1}}{(n+1)!} f^{(n+1)}(c),\ x<c<x+h, \text{ and}$$

c is some point in the domain $(x, x+h)$. So in this case, if we are using four terms of the Taylor series, the remainder is given by $(x=0, n=3)$

$$R_3(0+1)=\frac{(1)^{3+1}}{(3+1)!} f^{(3+1)}(c)$$
$$=\frac{1}{4!} f^{(4)}(c)$$
$$=\frac{e^c}{24}$$

Since

$$x<c<x+h$$
$$0<c<0+1$$
$$0<c<1$$

The error is bound between

$$\frac{e^0}{24}<R_3(1)<\frac{e^1}{24}$$
$$\frac{1}{24}<R_3(1)<\frac{e}{24}$$
$$0.041667<R_3(1)<0.113261$$

So the bound of the error is less than 0.113261 which does concur with the calculated error of 0.0516152.

Example 6

The Taylor series for e^x at point $x=0$ is given by

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots$$

As you can see in the previous example that by taking more terms, the error bounds decrease and hence you have a better estimate of e^1. How many terms it would require to get an approximation of e^1 within a magnitude of true error of less than 10^{-6}?

Solution

Using $(n+1)$ terms of the Taylor series gives an error bound of

$$R_n(x+h) = \frac{(h)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

$$x = 0, h = 1, f(x) = e^x$$

$$R_n(1) = \frac{(1)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

$$= \frac{(1)^{n+1}}{(n+1)!} e^c$$

Since

$$x < c < x + h$$

$$0 < c < 0 + 1$$

$$0 < c < 1$$

$$\frac{1}{(n+1)!} < |R_n(1)| < \frac{e}{(n+1)!}$$

So if we want to find out how many terms it would require to get an approximation of e^1 within a magnitude of true error of less than 10^{-6},

$$\frac{e}{(n+1)!} < 10^{-6}$$

$$(n+1)! > 10^6 e$$

$$(n+1)! > 10^6 \times 3$$ (as we do not know the value of e but it is less than 3).

$$n \geq 9$$

So 9 terms or more will get e^1 within an error of 10^{-6} in its value.

We can do calculations such as the ones given above only for simple functions. To do a similar analysis of how many terms of the series are needed for a specified accuracy for any general function, we can do that based on the concept of absolute relative approximate errors discussed in Chapter 01.02 as follows.

We use the concept of absolute relative approximate error (see Chapter 01.02 for details), which is calculated after each term in the series is added. The maximum value of m, for which the absolute relative approximate error is less than $0.5 \times 10^{2-m}$% is the least number of significant digits correct in the answer. It establishes the accuracy of the approximate value of a function without the knowledge of remainder of Taylor series or the true error.

Indeterminate Form

I. Indeterminate Form of the Type $\frac{0}{0}$

We have previously studied limits with the indeterminate form $\frac{0}{0}$ as shown in the following examples:

Example 1: $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x \to 2}(x+2) = 2 + 2 = 4$

Example 2: $\lim_{x\to 0} \dfrac{\tan 3x}{\sin 2x} = \lim_{x\to 0} \dfrac{\dfrac{\sin 3x}{\cos 3x}}{\sin 2x} = \lim_{x\to 0} \dfrac{\sin 3x}{1} \cdot \dfrac{1}{\cos 3x} \cdot \dfrac{1}{\sin 2x} =$

$$\frac{3}{2}\left(\lim_{3x\to 0} \frac{\sin 3x}{3x}\right)\left(\lim_{x\to 0} \frac{1}{\cos 3x}\right)\left(\lim_{2x\to 0} \frac{2x}{\sin 2x}\right) = \frac{3}{2}(1)(1)(1) = \frac{3}{2}$$

[*Note*: We use the given limit $\lim_{\Delta\to 0} \dfrac{\sin\Delta}{\Delta} = 1$.]

Example 3: $\lim_{h\to 0} \dfrac{\sqrt[3]{8+h} - 2}{h} = f'(8) = \dfrac{1}{3\sqrt[3]{8^2}} = \dfrac{1}{12}$. [*Note*: We use the definition of the derivative $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ where $f(x) = \sqrt[3]{x}$ and a = 8.]

Example 4: $\lim_{x\to \pi/3} \dfrac{\cos x - 1/2}{x - \pi/3} = f'(\pi/3) = -\sin(\pi/3) = -\sqrt{3}/2$. [*Note*: We use the definition of the derivative $f'(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x-a}$ where $f(x) = \cos x$ and $a = \pi/3$.]

However, there is a general, systematic method for determining limits with the indeterminate form $\dfrac{0}{0}$. Suppose that *f* and *g* are differentiable functions at *x* = *a* and that $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is an indeterminate form of the type $\dfrac{0}{0}$; that is, $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Since *f* and *g* are differentiable functions at *x* = *a*, then *f* and *g* are continuous at *x* = *a*; that is, $f(a) = \lim_{x\to a} f(x) = 0$ and $g(a) = \lim_{x\to a} g(x) = 0$. Furthermore, since *f* and *g* are differentiable functions at *x* = *a*, then $f'(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x-a}$ and $g'(a) = \lim_{x\to a} \dfrac{g(x) - g(a)}{x-a}$. Thus, if $g'(a) \neq 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x\to a} \frac{\dfrac{f(x) - f(a)}{x-a}}{\dfrac{g(x) - g(a)}{x-a}} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$ if f' and g' are continuous at *x* = *a*. This illustrates a special case of the technique known as

L'Hospital's Rule.

L'Hospital's Rule for Form $\frac{0}{0}$

Suppose that *f* and *g* are differentiable functions on an open interval containing *x* = *a*, except possibly at

$x = a$, and that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ has a finite limit, or if this limit is $+\infty$ or $-\infty$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$. Moreover, this statement is also true in the case of a limit as $x \to a^-, x \to a^+, x \to -\infty,$ or as $x \to +\infty$.

In the following examples, we will use the following three-step process:

Step 1. Check that the limit of $\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$**.** If it is not, then **L'Hospital's Rule** cannot be used.

Step 2. Differentiate *f* and *g* separately. [*Note*: **Do not differentiate** $\frac{f(x)}{g(x)}$ **using the quotient rule!**]

Step 3. Find the limit of $\frac{f'(x)}{g'(x)}$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $\frac{f(x)}{g(x)}$**.** If the limit is an indeterminate form of type $\frac{0}{0}$**,** then simplify $\frac{f'(x)}{g'(x)}$ algebraically and apply **L'Hospital's Rule** again.

Example 1: $\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{2x}{1} = 2(2) = 4$

Example 2: $\lim_{x\to 0}\frac{\tan 3x}{\sin 2x} = \lim_{x\to 0}\frac{3\sec^2 3x}{2\cos 2x} = \frac{3(1)}{2(1)} = \frac{3}{2}$

Example 3: $\lim_{h\to 0}\frac{\sqrt[3]{8+h}-2}{h} = \lim_{h\to 0}\frac{\frac{1}{3}(8+h)^{-2/3}(1)}{1} = \lim_{h\to 0}\frac{1}{3(8+h)^{2/3}} = \frac{1}{3(8)^{2/3}} = \frac{1}{12}$

Example 4: $\lim_{x\to \pi/3}\frac{\cos x - 1/2}{x - \pi/3} = \lim_{x\to \pi/3}\frac{-\sin x}{1} = -\sin(\pi/3) = -\frac{\sqrt{3}}{2}$

Example 5: $\lim_{x\to 0}\frac{e^x - x - 1}{x^2} = \lim_{x\to 0}\frac{e^x - 1}{2x} = \lim_{x\to 0}\frac{e^x}{2} = \frac{1}{2}$ [Use **L'Hospital's Rule** twice.]

Example 6: $\lim_{x\to +\infty}\frac{1/x^2}{\sin(1/x)} = \lim_{x\to +\infty}\frac{-2/x^3}{\cos(1/x)(-1/x^2)} = \lim_{x\to +\infty}\frac{2/x}{\cos(1/x)} = \frac{0}{1} = 0$, or

$$\lim_{x\to+\infty}\frac{1/x^2}{\sin\left(1/x\right)}=\lim_{y\to0^+}\frac{y^2}{\sin y}=\lim_{y\to0^+}\frac{2y}{\cos y}=\frac{2(0)}{1}=0 \text{ where } y=1/x.$$

Example 7: $\lim_{x\to0}\frac{x}{\ln x}=\lim_{x\to0} x\left(1/\ln x\right)=0(0)=0$ [This limit is **not** an indeterminate form of the type $\frac{0}{0}$, so **L'Hospital's Rule** cannot be used.]

II. Indeterminate Form of the Type $\frac{\infty}{\infty}$

We have previously studied limits with the indeterminate form $\frac{\infty}{\infty}$ as shown in the following examples:

Example 1: $$\lim_{x\to+\infty}\frac{3x^2+5x-7}{2x^2-3x+1}=\lim_{x\to+\infty}\frac{\frac{3x^2}{x^2}+\frac{5x}{x^2}-\frac{7}{x^2}}{\frac{2x^2}{x^2}-\frac{3x}{x^2}+\frac{1}{x^2}}=\lim_{x\to+\infty}\frac{3+\frac{5}{x}-\frac{7}{x^2}}{2-\frac{3}{x}+\frac{1}{x^2}}=\lim_{x\to+\infty}\frac{3+0-0}{2-0+0}=\frac{3}{2}$$

Example 2: $$\lim_{x\to-\infty}\frac{3x-1}{x^2+1}=\lim_{x\to-\infty}\frac{\frac{3x}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}}=\lim_{x\to-\infty}\frac{\frac{3}{x}-\frac{1}{x^2}}{1+\frac{1}{x^2}}=\frac{0-0}{1+0}=\frac{0}{1}=0$$

Example 3: $$\lim_{x\to\infty}\frac{3x^3-4}{2x^2+1}=\lim_{x\to\infty}\frac{\frac{3x^3}{x^3}-\frac{4}{x^3}}{\frac{2x^2}{x^3}+\frac{1}{x^3}}=\lim_{x\to\infty}\frac{3-\frac{4}{x^3}}{\frac{2}{x}+\frac{1}{x^3}}=\frac{3-0}{0+0}=\frac{3}{0}\Rightarrow \quad \text{limit does not exist.}$$

Example 4: $$\lim_{x\to-\infty}\frac{\sqrt{4x^2+1}}{x+1}=\lim_{x\to-\infty}\frac{\frac{\sqrt{4x^2+1}}{x}}{\frac{x+1}{x}}=\lim_{x\to-\infty}\frac{\frac{\sqrt{4x^2+1}}{-\sqrt{x^2}}}{\frac{x+1}{x}}$$ ($\sqrt{x^2}=-x$ because $x<0$ and thus $x=-\sqrt{x^2}$) $$=\lim_{x\to-\infty}\frac{-\sqrt{\frac{4x^2+1}{x^2}}}{\frac{x+1}{x}}=\lim_{x\to-\infty}\frac{-\sqrt{4+\frac{1}{x^2}}}{1+\frac{1}{x^2}}=\frac{-\sqrt{4}}{1}=-2.$$

However, we could use another version of **L'Hospital's Rule**.

L'Hospital's Rule for Form $\frac{\infty}{\infty}$

Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. If $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ has a finite limit, or if this limit is $+\infty$ or $-\infty$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$. Moreover, this statement is also true in the case of a limit as $x \to a^-, x \to a^+, x \to -\infty,$ or as $x \to +\infty.$

Example 1: $\lim_{x\to+\infty}\frac{3x^2+5x-7}{2x^2-3x+1} = \lim_{x\to+\infty}\frac{6x+5}{4x-3} = \lim_{x\to+\infty}\frac{6}{4} = \frac{3}{2}$

Example 2: $\lim_{x\to-\infty}\frac{3x-1}{x^2+1} = \lim_{x\to-\infty}\frac{3}{2x} = \frac{3}{2}\lim_{x\to-\infty}\frac{1}{x} = \frac{3}{2}(0) = 0$

Example 3: $\lim_{x\to\infty}\frac{3x^3-4}{2x^2+1} = \lim_{x\to\infty}\frac{9x^2}{4x} = \lim_{x\to\infty}\frac{18x}{4} = \infty$

Example 4: $\lim_{x\to-\infty}\frac{\sqrt{4x^2+1}}{x+1} = \lim_{x\to-\infty}\frac{\frac{8x}{2\sqrt{4x^2+1}}}{1} = \lim_{x\to-\infty}\frac{4x}{\sqrt{4x^2+1}} \Rightarrow$ **L'Hospital's Rule** does not help in this situation. We would find the limit as we did previously.

Example 5: $\lim_{x\to+\infty}\frac{\ln(x^2+1)}{\ln(x^3+1)} = \lim_{x\to+\infty}\frac{\frac{2x}{x^2+1}}{\frac{3x^2}{x^3+1}} = \lim_{x\to+\infty}\frac{2x(x^3+1)}{3x^2(x^2+1)} = \lim_{x\to+\infty}\frac{2x^4+2x}{3x^4+3x^2} =$

$$\lim_{x\to+\infty}\frac{8x^3+2}{12x^3+6x} = \lim_{x\to+\infty}\frac{24x^2}{36x^2+6} = \lim_{x\to+\infty}\frac{48x}{72x} = \frac{48}{72} = \frac{2}{3}$$

Example 6: $\lim_{x\to0^+}\frac{\ln x}{1/x^2} = \lim_{x\to0^+}\frac{1/x}{-2/x^3} = \lim_{x\to0^+}\frac{x^3}{-2x} = \lim_{x\to0^+}\frac{x^2}{-2} = \frac{0^2}{-2} = 0$

Example 7: $\lim\limits_{x\to+\infty} \frac{\arctan x}{x} = \left(\lim\limits_{x\to+\infty} \frac{1}{x}\right)\left(\lim\limits_{x\to+\infty} \arctan x\right) = (0)\left(\frac{\pi}{2}\right) = 0$ [This limit is **not** an indeterminate form of the type $\frac{\infty}{\infty}$, so **L'Hospital's Rule** cannot be used.]

III. Indeterminate Form of the Type $0\cdot\infty$

Indeterminate forms of the type $0\cdot\infty$ can sometimes be evaluated by rewriting the product as a quotient, and then applying **L'Hospital's Rule** for the indeterminate forms of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 1: $\lim\limits_{x\to0^+} x\ln x = \lim\limits_{x\to0^+} \frac{\ln x}{1/x} = \lim\limits_{x\to0^+} \frac{1/x}{-1/x^2} = \lim\limits_{x\to0^+} \frac{-x^2}{x} = \lim\limits_{x\to0^+}(-x) = 0$

Example 2: $\lim\limits_{x\to0^+} (\sin x)\ln x = \lim\limits_{x\to0^+} \frac{\ln x}{\csc x} = \lim\limits_{x\to0^+} \frac{1/x}{-\csc x\cot x} = \lim\limits_{x\to0^+} \frac{-\sin x\tan x}{x} =$

$$\left(\lim_{x\to0^+} \frac{-\sin x}{x}\right)\left(\lim_{x\to0^+} \tan x\right) = (-1)(0) = 0$$

Example 3: $\lim\limits_{x\to+\infty} x\sin\left(1/x\right) = \lim\limits_{x\to+\infty} \frac{\sin\left(1/x\right)}{1/x} = \lim\limits_{y\to0^+} \frac{\sin y}{y} = 1$ [Let $y = 1/x$.]

IV. Indeterminate Form of the Type $\infty-\infty$

A limit problem that leads to one of the expressions

$$(+\infty)-(+\infty),\quad (-\infty)-(-\infty),\quad (+\infty)+(-\infty),\quad (-\infty)+(+\infty)$$

is called an **indeterminate form of type** $\infty-\infty$. Such limits are indeterminate because the two terms exert conflicting influences on the expression; one pushes it in the positive direction and the other pushes it in the negative direction. However, limits problems that lead to one the expressions

$$(+\infty)+(+\infty),\quad (+\infty)-(-\infty),\quad (-\infty)+(-\infty),\quad (-\infty)-(+\infty)$$

are not indeterminate, since the two terms work together (the first two produce a limit of $+\infty$ and the last two produce a limit of $-\infty$). Indeterminate forms of the type $\infty-\infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 1: $\lim\limits_{x\to0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim\limits_{x\to0^+} \left(\frac{\sin x - x}{x\sin x}\right) = \lim\limits_{x\to0^+} \frac{\cos x - 1}{x\cos x + \sin x} =$

$$\lim_{x\to 0^+} \frac{-\sin x}{-x\sin x+\cos x+\cos x} = \frac{0}{2} = 0$$

Example 2: $\lim_{x\to 0} \left[\ln(1-\cos x) - \ln\left(x^2\right)\right] = \lim_{x\to 0} \left[\ln\left(\frac{1-\cos x}{x^2}\right)\right] =$

$$\ln\left[\lim_{x\to 0} \left(\frac{1-\cos x}{x^2}\right)\right] = \ln\left[\lim_{x\to 0} \left(\frac{\sin x}{2x}\right)\right] = \ln\left(\frac{1}{2}\right)$$

V. Indeterminate Forms of the Types $0^0, \infty^0, 1^\infty$

Limits of the form $\lim_{x\to a} [f(x)]^{g(x)}$ $\left\{or \ \lim_{x\to\infty} [f(x)]^{g(x)}\right\}$ frequently give rise to indeterminate forms of the types $0^0, \infty^0, 1^\infty$. These indeterminate forms can sometimes be evaluated as follows:

(1) $y = [f(x)]^{g(x)}$

(2) $\ln y = \ln[f(x)]^{g(x)} = g(x)\ln[f(x)]$

(3) $\lim_{x\to a} [\ln y] = \lim_{x\to a} \{g(x)\ln[f(x)]\}$

The limit on the righthand side of the equation will usually be an indeterminate limit of the type $0\cdot\infty$. Evaluate this limit using the technique previously described. Assume that $\lim_{x\to a} \{g(x)\ln[f(x)]\} = L$.

(4) Finally, $\lim_{x\to a} [\ln y] = L \Rightarrow \ln\left[\lim_{x\to a} y\right] = L \Rightarrow \lim_{x\to a} y = e^L$.

Example 1: Find $\lim_{x\to 0^+} x^x$.

This is an indeterminate form of the type 0^0. Let $y = x^x \Rightarrow \ln y = \ln x^x =$ $x\ln x$. $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0.$

Thus, $\lim_{x\to 0^+} x^x = e^0 = 1$.

Example 2: Find $\lim_{x\to +\infty} (e^x+1)^{-2/x}$.

This is an indeterminate form of the type ∞^0. Let $y = (e^x+1)^{-2/x} \Rightarrow$ $\ln y = \ln\left[(e^x+1)^{-2/x}\right] = \frac{-2\ln(e^x+1)}{x}$. $\lim_{x\to +\infty} \ln y = \lim_{x\to +\infty} \frac{-2\ln(e^x+1)}{x} =$

$$\lim_{x\to +\infty} \frac{-2\left(\frac{e^x}{e^x+1}\right)}{1} = \lim_{x\to +\infty} \frac{-2e^x}{e^x+1} = \lim_{x\to +\infty} \frac{-2e^x}{e^x} = -2$$

Thus, $\lim_{x\to +\infty} (e^x+1)^{-2/x} =$

e^{-2}.

Example 3: Find $\lim_{x\to 0^+} (\cos x)^{1/x}$.

This is an indeterminate form of the type 1^{∞}. Let $y = (\cos x)^{1/x} \Rightarrow$
$\ln y = \ln\left[(\cos x)^{1/x}\right] = \frac{\ln(\cos x)}{x}$. $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(\cos x)}{x} =$
$\lim_{x\to 0^+}(-\tan x) = 0$. Thus, $\lim_{x\to 0^+} (\cos x)^{1/x} = e^0 = 1$.

Tangents

The tangent to the graph of a function f at the point $(c, f(c))$ is a line such that:

- its slope is equal to $f'(c)$.
- it passes through the point $(c, f(c))$.

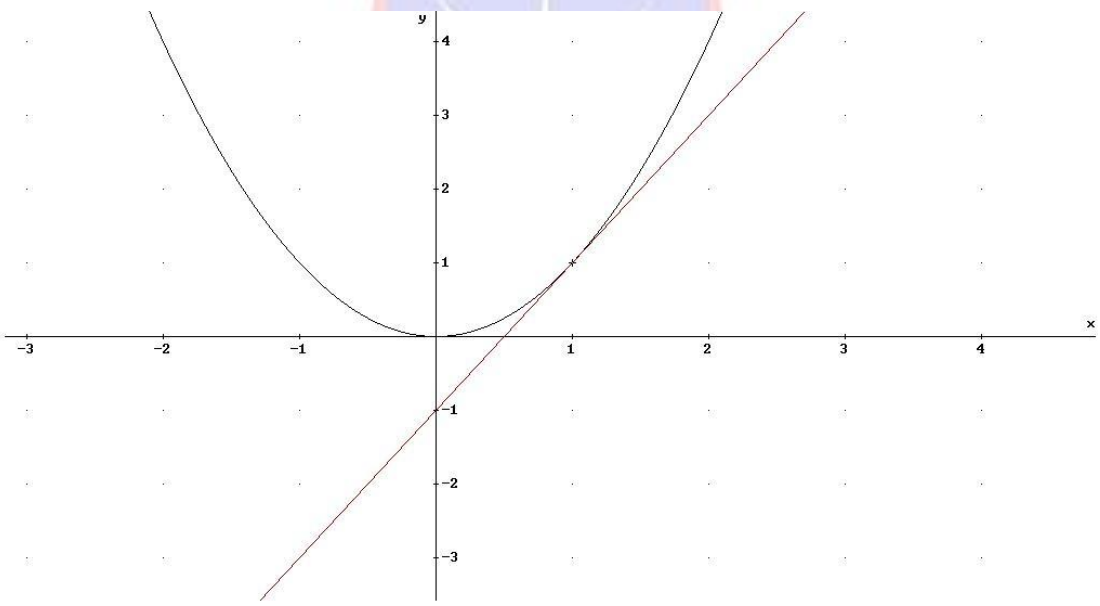
The equation of the tangent to the graph of a function f at the point $(c, f(c))$ is given by the following formula:

$$y = f'(x)(x-c) + f(x).$$

<u>Example</u>: Find the equation of the tangent to the graph of $f(x) = x^2$ at the point $(1,1)$.

We have $f'(x) = 2x$ and, since $c = 1$, we obtain

$y = f'(1)(x-1) + f(1) \quad \Rightarrow \quad y = 2(x-1) + 1 \quad \Rightarrow \quad y = 2x - 1.$



Maximum and minimum

A function $f(x)$ is said to have a **local maximum** at x_0 if there exists $a > 0$ such that, for $x \in (x_0 - a, x_0 + a)$, we have $f(x) \le f(x_0)$.

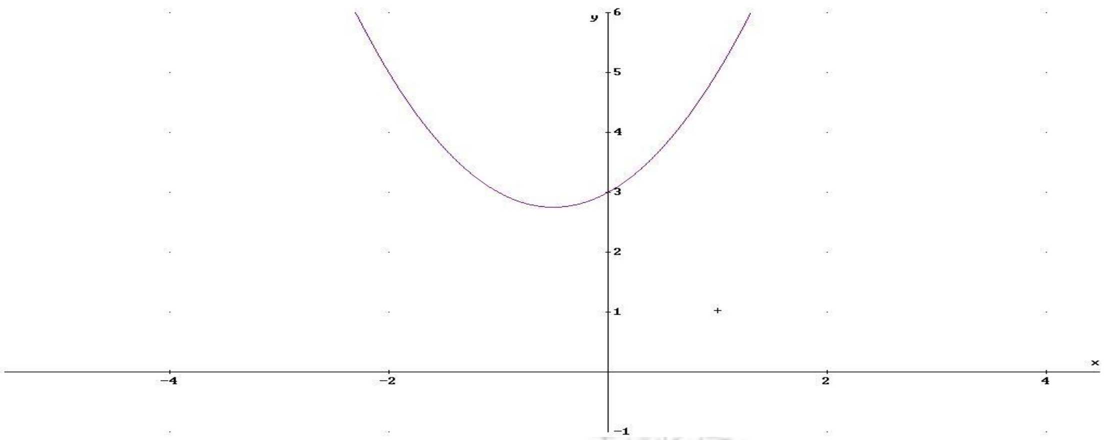
Intuitively, it means that around x_0 the graph of f will be below $f(x_0)$.

Similarly, a function $f(x)$ is said to have a **local minimum** at x_0 if there exists $a > 0$ such that, for $x \in (x_0 - a, x_0 + a)$, we have $f(x) \ge f(x_0)$.

This time, the graph of f will be situated above $f(x_0)$ for values of x around x_0.

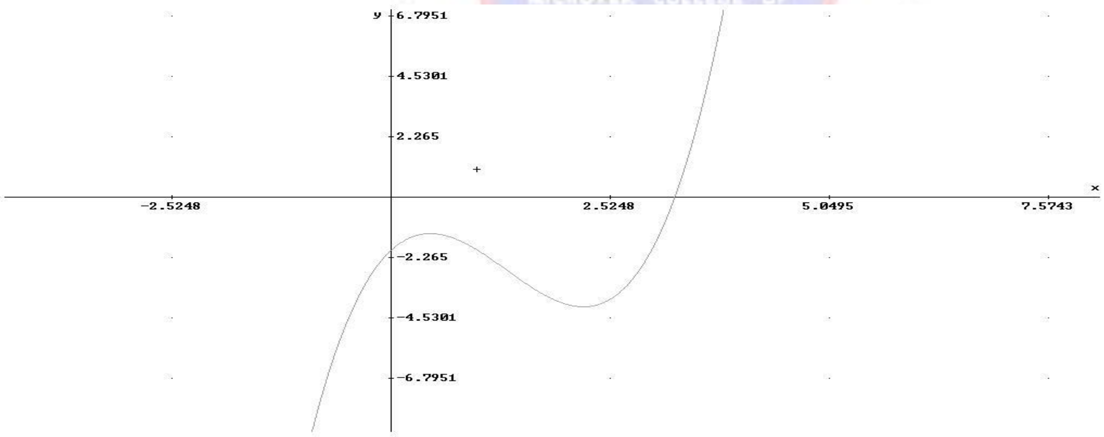
<u>Examples</u>:

$f(x) = x^2 + x + 3$.



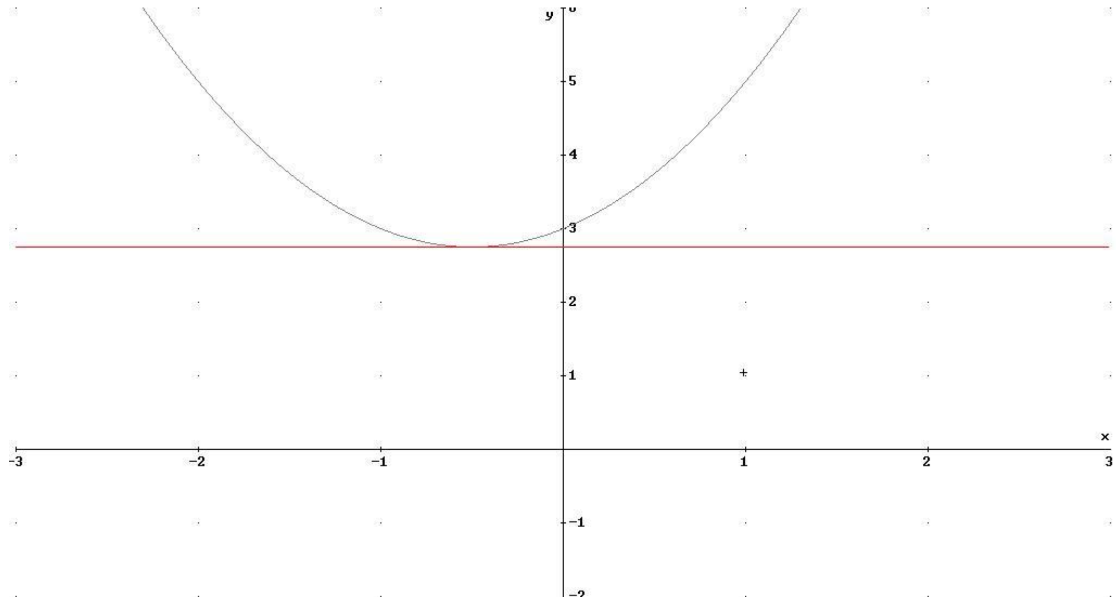
From the graph, it is rather obvious that the function has a unique minimum and that this minimum is global (i.e. the whole graph is above this minimum).

On the other hand, if we take $f(x) = x^3 - 4x^2 + 3x - 2$, the situation is rather different:



Here, we have a local maximum and a local minimum.

Minima and maxima have one thing in common: say f has a local minimum at x_0. Then the tangent to the graph of f at the point $(x_0, f(x_0))$ is a horizontal line:



The slope of the tangent is therefore 0.
Remember, the slope of the tangent to the graph of f at the point $\left(x_0, f(x_0)\right)$ is equal to $f'(x_0)$, so here we end up with $f'(x_0) = 0$.

If f has a local minimum or a local maximum at x_0, we therefore have $f'(x_0) = 0$.

In general, the solutions of $f'(x) = 0$ are called **stationary points**. There are three different kinds of stationary points: **local minima, local maxima and turning points**.

You can classify them as follows:

Say x_0 is a stationary point. Then if

- $f''(x_0) < 0$, there is a local maximum at x_0.
- $f''(x_0) > 0$, there is a local minimum at x_0.
- $f''(x_0) = 0$, there is a turning point at x_0.

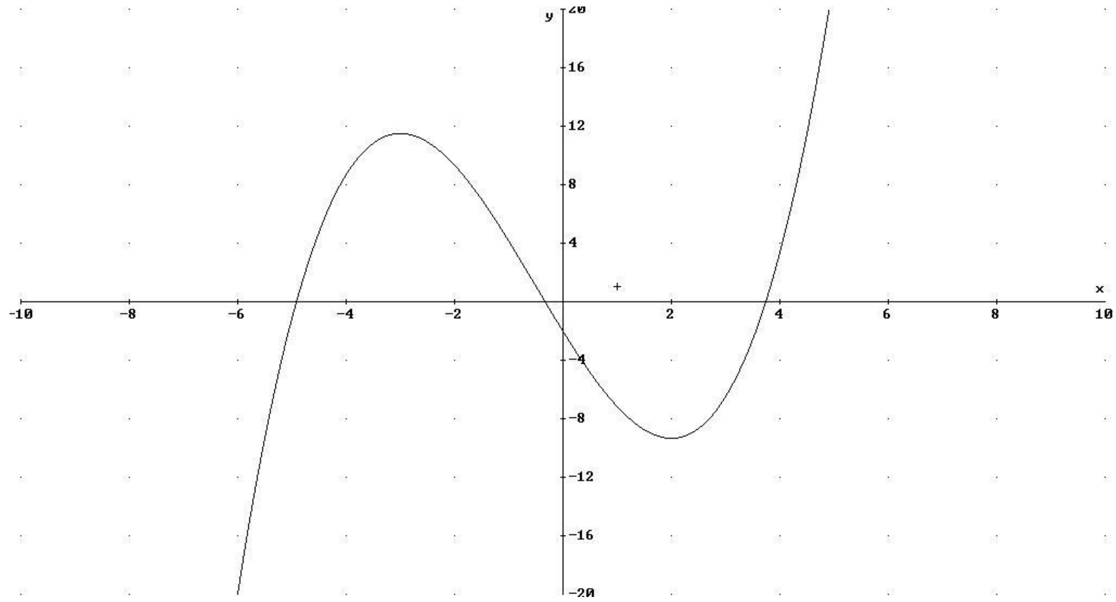
<u>Example</u>: $f(x) = \dfrac{x^3}{3} + \dfrac{x^2}{2} - 6x - 2$. Find and classify the stationary points of f .To find the stationary points, we solve $f'(x) = 0$:

Here, $f'(x) = x^2 + x - 6 = (x-2)(x+3)$, so that $f'(x) = 0 \quad \Leftrightarrow \quad x = 2 \quad or \quad x = -3$.

Next, we calculate $f''(x)$ and use the rule above to classify the stationary points:

$f''(x) = 2x + 1$.
$f''(2) = 5 > 0$, so that f has a local minimum at $x = 2$.
$f''(-3) = -5 < 0$, so that f has a local maximum at $x = -3$.

Let's have a look at the graph of f :



The graph indicates that there is indeed a local minimum at $x=2$ and a local maximum at $x=-3$. The graph also indicates that they are both local and not global.

Successive Differentiation:

The derivative f' (x) of a derivable function f (x) is itself a function of x. We suppose that it also possesses a derivative, which is denoted by f'' (x) and called the second derivative of f (x). The third derivative of f (x) which is the derivative of f'' (x) is denoted by f '''(x) and so on. Thus the successive derivatives of f (x) are represented by the symbols, f (x), f; (x), . . . , f^n (x), . . .
where each term is the derivative of the previous one. Sometimes y_1 , y_2 , y_3 , . . . , y_n , . . . are used to denote the successive derivatives of y.

• Leibnitz's Theorem

The nth derivative of the product of two functions: If u, v be the two functions possessing derivatives of the nth order, then $(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots + {}^nC_r u_{n-r} v_r + \dots + uv_n$.

UNIT - IV
INTEGRAL

Contents

INDEFINITE INTEGRATION

Definition $f(x)$ is said to be **primitive** function or **anti-derivative** of $g(x)$ if $f'(x) = g(x)$.

Example $\frac{d}{dx}(x^2) = 2x$ $\therefore$ x^2 is the primitive function of $2x$.

Note Primitive function is not **UNIQUE**.

Definition For any function $f(x)$ if $F(x)$ is the primitive function of $f(x)$, i.e. $F'(x) = f(x)$, then we define the indefinite integral of $f(x)$ w.r.t.x as $\int f(x)\,dx = F(x) + c$, where c is called **the constant of integration**.

Theorem Two function $f(x)$ and $h(x)$ differ by a constant if and only if they have the same primitive function.

Standard Results

1. $\int \frac{1}{x}dx = \ln x + c$
2. $\int e^x dx = e^x + c$
3. $\int \cos x dx = \sin x + c$
4. $\int \sin x dx = -\cos x + c$
5. $\int \sec^2 x dx = \tan x + c$
6. $\int \csc^2 x dx = -\cot x + c$
7. $\int \sec x \tan x dx = \sec x + c$
8. $\int \csc x \cot x dx = -\csc x + c$
9. $\int a^x dx = \frac{a^x}{\ln a} + c$
10. $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + c$

11*. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a} + c$

12*. $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

13. $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| + c$

Theorem (a) $\int kf(x)dx = k\int f(x)dx$

(b) $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$.

Example Prove $\displaystyle\int a^x dx = \frac{a^x}{\ln a} + c$

proof Let $y = a^x$.

$\ln y = x\ln a \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \ln a \quad \therefore \quad \frac{dy}{dx} = y\ln a$

$$\int \frac{dy}{dx}dx = \int y\ln a\, dx$$

$$y = \ln a \int y\, dx$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

METHOD OF SUBSTITUTION

Theorem **(CHANGE OF VARIABLE)**

If $x = g(t)$ is a differentiable function, $\int f(x)dx = \int f(g(t))\,g'(t)\,dt$.

Proof Let $F(x)$ is the primitive function of $f(x)$.

i.e. $\displaystyle\frac{dF(x)}{dx} = f(x)$

$\because$ $x = g(t)$

We have
$$\frac{d}{dt}F(x) = \frac{dF(x)}{dx}\cdot\frac{dx}{dt}$$
$$= \frac{dF(x)}{dx}g'(t)$$
$$F(x) = \int f(g(t))g'(t)dt$$
$$\int f(x)dx = \int f(g(t))g'(t)dt$$

Example Prove $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| + c$

proof sub $x = a\tan\theta \quad \Rightarrow \quad dx = a\sec^2\theta\, d\theta$

$$\therefore \int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{1}{a\sec\theta}a\sec^2\theta\, d\theta$$
$$= \int \sec\theta\, d\theta$$
$$= \ln|\sec\theta + \tan\theta| + c$$

($\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + c$)

$$= \ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| + c$$

Remark By using substitution, the following two formulae can be derived easily.

(I) $$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c,$$

(II) $$\int \frac{f'(x)}{2\sqrt{f(x)}}dx = \sqrt{f(x)} + c.$$

The following examples illustrate the use of the above results.

Example $\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + c$ and $\int \csc\theta \, d\theta = -\ln|\csc\theta + \cot\theta| + c$

proof

Example $\int \tan\theta \, d\theta$ $\qquad$ $\int \frac{\ln x}{x}dx$

$$= \int \frac{\sin\theta}{\cos\theta}d\theta \qquad = \int \ln x \, d(\ln x)$$

$$= -\int \frac{1}{\cos\theta}d(\cos\theta) \qquad = \frac{(\ln x)^2}{2} + c$$

$$= -\ln|\cos\theta| + c$$

Example $\int \cot\theta \, d\theta$ $\qquad$ $\int \frac{\cos x}{3 + 2\sin x}dx$

$$= \int \frac{\cos\theta}{\sin\theta}d\theta$$

$$= \int \frac{1}{\sin\theta}d(\sin\theta)$$

$$= \ln|\sin\theta| + c$$

Example **(a)** $\int \frac{dx}{e^x}$ **(b)** $\int x^2 e^{x^3} dx$

Example $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$ (Let $y = \sqrt{x}$)

Example (a) $\int \frac{e^x + 1}{e^x - 1}dx$ (b) $\int \frac{e^{\sin 2x}\sin^2 x}{e^{2x}}dx$

Example $\int x^2\sqrt{1+x}\,dx$

Example (a) $\int e^x \cot(e^x)dx$

(b) $\int \frac{a^{2x} + b^{2x}}{e^x}dx$

(c) $\int \frac{x^2}{px-q}dx$

INTEGRATION BY PARTS

***Theorem** **(INTEGRATION BY PARTS)**

If u, v are two functions of x, then $\int udv = uv - \int vdu.$

proof

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$u\frac{dv}{dx} = \frac{d}{dx}uv - v\frac{du}{dx}$$

We integrate both sides with respect to x to obtain

$$\int udv = \int u\frac{dv}{dx}dx = uv - \int vdu$$

Example (a) $\int \ln x\, dx$ (b) $\int x^2 \ln x\, dx$

Example (a) $I = \int x^2 \cos x dx$

(b) $I = \int x^2 \sin x dx$

Example $I = \int \frac{xe^x}{(1+x)^2}dx$

Example $\int \tan^{-1} x\, dx$

Example $\int (\ln x)^2 dx$

Example (a) Show that $\frac{d}{dx}\tan\frac{x}{2} = \frac{1}{1+\cos x}$.

(b) Using (a), or otherwise, find $\int \frac{x+\sin x}{1+\cos x}dx$

SPECIAL INTEGRATION

We resolve the rational function $\frac{P(x)}{Q(x)}$ by simple partial fraction for $P(x), Q(x)$ being poly. The integration of rational function is easily done by terms by terms integration.

Example (a) $\int \frac{dx}{x^2 - a^2}$ (b) $\int \frac{x+1}{x^2+1}dx$

Example $\int \frac{x^3 + 2x^2 + 1}{(x-1)(x-2)(x-3)^2}dx$

Example Evaluate $\int \frac{2x^4 + x^3 + 3x^2 - 3x}{x^3 - 1}dx$.

Solution By decomposing into partial fractions,

$$\frac{2x^4 + x^3 + 3x^2 - 3x}{x^3 - 1} = 2x + 1 + \frac{1}{x-1} + \frac{2x}{x^2 + x + 1}.$$

Hence,

Integration of $\frac{Px + Q}{\sqrt{ax^2 + bx + c}}$

Example Evaluate $\int \frac{4x-1}{\sqrt{5 - 4x - x^2}}dx.$

Solution Observing that the derivative of $5 - 4x - x^2$ is $-(4+2x)$, we have

$$\int \frac{4x-1}{\sqrt{5-4x-x^2}}dx = \int \frac{-2(-4-2x)-9}{\sqrt{5-4x-x^2}}dx$$

Integration of $\frac{1}{x \pm \sqrt{ax^2 + bx + c}}$

Example $\int \frac{dx}{x - \sqrt{x^2 - 1}}$

Integration of $\int R(x, \sqrt[n]{\frac{ax+b}{cx+d}})dx$

In solving such problems, we use the substitution $u = \sqrt[n]{\frac{ax+b}{cx+d}}$

Example $I = \int \frac{x+2}{x\sqrt{x+1}}dx$

INTEGRATION OF TRIGONOMETRIC FUNCTION

Integration of $\int R(\cos\theta, \sin\theta)\,d\theta$

(1) If $-R(-\cos\theta, \sin\theta) = R(\cos\theta, \sin\theta)$, put $u = \sin\theta$.

(2) If $-R(\cos\theta, -\sin\theta) = R(\cos\theta, \sin\theta)$, put $u = \cos\theta$.

(3) If $R(-\cos\theta, -\sin\theta) = R(\cos\theta, \sin\theta)$, put $u = \tan\theta$.

(4) Otherwise, put $t = \tan\frac{\theta}{2}$. $\therefore$

$$\tan\theta = \frac{2t}{1-t^2}$$
$$\cos\theta = \frac{1-t^2}{1+t^2}$$
$$\sin\theta = \frac{2t}{1+t^2}$$

Example (a) $\int \cos^3\theta \sin^2\theta \, d\theta$ (b) $\int \cos^2\theta \sin^3\theta \, d\theta$

REDUCTION FORMULA

Certain integrals involving powers of the variable or powers of functions of the variable can be related to integrals of the same form but containing reduced powers and such relations are called **REDUCTION FORMULAS (**Successive use of such formulas will often allow a given integral to be expressed in terms of a much simpler one.

Example Let $I_n = \int \sin^n x \, dx$ for n is non-negative integer.

Show that $I_n = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}$

Hence, find I_6 .

Example Show that if $I_n = \int \cos^n\theta \, d\theta$, where n is a non-negative integer, then

$$I_n = \frac{\sin\theta \cos^{n-1}\theta}{n} + \frac{n-1}{n} I_{n-2}, \text{ for } n \geq 2.$$

Hence evaluate I_5 and I_6 .

Example If $I_n = \int \tan^n x \, dx$, where n is a non-negative integer, find a reduction formula for I_n .

($I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$)

This formula relates I_n with I_{n-2}, and if **n** is a positive integer, successive use of it will ultimately relate with either $\int \tan x \, dx$ or $\int dx$ .Since $\int \tan x dx = \ln|\sec x| + c, \int dx = x + c$, and positive integral power of $\tan x$ can therefore be integrated.

Example For non-negative integer **n**, $\mathbf{I_n = \int (\ln x)^n dx}$.

Find a reduction formula for $\mathbf{I_n}$ and hence evaluate $\mathbf{I_3}$.

Example Let **n** be a positive integer and $\mathbf{a \neq 0}$.

$$\mathbf{I_n = \int \frac{dx}{(ax^2+bx+c)^n}} \qquad \cdots(*)$$

(a) Prove that $\mathbf{n(4ac-b^2)I_{n+1} = 2(2n-1)a\, I_n + \frac{2ax+b}{(ax^2+bx+c)^n}}$.

(b) Evaluate $\mathbf{\int \frac{dx}{(x^2-2x+2)^2}}$.

METHODS OF INTGRATION

1. **Integration using formulae i.e. simple integration**
2. **Integration by substitution**

(i) Integrand of the form $f(ax+b)$

FORMULAE BASED ON $f(ax+b)$

1. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, n \neq -1$

2. $\int \frac{1}{ax+b} dx = \frac{\log|ax+b|}{a} + c$

3. $\int c^{ax+b} dx = \frac{c^{ax+b}}{a \log c} + k$

4. $\int e^{ax+b} dx = \frac{e^{ax+b}}{a} + c$

5. $\int \sin(ax+b) dx = -\frac{\cos(ax+b)}{a} + c$

6. $\int \cos(ax+b) dx = \frac{\sin(ax+b)}{a} + c$

7. $\int \sec^2(ax+b)\ dx = \frac{\tan(ax+b)}{a} + c$

8. $\int \operatorname{cosec}^2(ax+b)\ \mathrm{dx} = -\frac{\cot(ax+b)}{\mathrm{a}} + \mathrm{c}$

9. $\int \sec(ax+b)\tan(ax+b) dx = \frac{\sec(ax+b)}{a} + c$

10. $\int \operatorname{cosec}(ax+b)\cot(ax+b) dx = -\frac{\operatorname{cosec}(ax+b)}{a} + c$

11. $\int \tan(ax+b) dx = -\frac{\log\cos|ax+b|}{a} + c$ **or** $\frac{\log\sec|ax+b|}{a} + c$

12. $\int \cot(ax+b) dx = \frac{\log\sin|ax+b|}{a} + c$

13. $\int \sec(ax+b) dx = \frac{\log|\sec(ax+b)+\tan(ax+b)|}{a} + c$ or $\frac{\log\left|\tan\left(\frac{\pi}{4}+\frac{(ax+b)}{2}\right)\right|}{a} + c$

14. $\int \operatorname{cosec}(ax+b)\,dx = \dfrac{\log\left|\operatorname{cosec}(ax+b)-\cot(ax+b)\right|}{a}+c$ **or** $\dfrac{\log\left|\tan\dfrac{(ax+b)}{2}\right|}{a}+c$

15. $\int \dfrac{1}{\sqrt{1-(ax+b)^2}}dx = \dfrac{\sin^{-1}(ax+b)}{a} + c$

16. $\int -\dfrac{1}{\sqrt{1-(ax+b)^2}}dx = \dfrac{\cos^{-1}(ax+b)}{a} + c$

17. $\int \dfrac{1}{1+(ax+b)^2}\,dx = \dfrac{\tan^{-1}(ax+b)}{a}+c$

18. $\int -\dfrac{1}{1+(ax+b)^2}\,dx = \dfrac{\cot^{-1}(ax+b)}{a}+c$

19. $\int \dfrac{1}{(ax+b)\sqrt{(ax+b)^2-1}}\,dx = \dfrac{\sec^{-1}(ax+b)}{a}+c$

20. $\int -\dfrac{1}{(ax+b)\sqrt{(ax+b)^2-1}}dx = \dfrac{\operatorname{cosec}^{-1}(ax+b)}{a}+c$

(ii) Integration of the type $\int [f(x)]^n . f'(x)\,dx$ **;** $\int \dfrac{f'(x)}{[f(x)]^n}\,dx$ **;** $\int \dfrac{f'(x)}{f(x)}\,dx$ **;** $\int g(f(x)).f'(x)\,dx$

METHOD: Put f(x) = t and f'(x) dx = dt and proceed.

NOTE: $\int \dfrac{f'(x)}{f(x)}\,dx = \log|f(x)|+c$

(iii) Integration of the type: $\int \sin^m(x).\cos^n(x)dx$**, where either 'm' or 'n' or both are odd.**

METHOD:

Case(i) If power of sine i.e. m is odd and power of cosine i.e. n is even then put $\cos x = t$ **and proceed.**

Case(ii) If power of sine i.e. m is even and power of cosine i.e. n is odd then put $\sin x = t$ **and proceed.**

Case(iii) If power of sine i.e. m is odd and power of cosine i.e. n is also odd then put $\cos x = t$ **or** $\sin x = t$ **and proceed.**

(iv). Integration which requires simplification by trigonometric functions:

Learn the following formulae:

$\sin^2 x = \dfrac{1-\cos 2x}{2}$	$2\sin A\cos B = \sin(A+B)+\sin(A-B)$

$\cos^2 x = \dfrac{1+\cos 2x}{2}$	$2\cos A\sin B = \sin(A+B) - \sin(A-B)$
$\sin^3 x = \dfrac{1}{4}[3\sin x - \sin 3x]$	$2\cos A\cos B = \cos(A+B) + \cos(A-B)$
$\cos^3 x = \dfrac{1}{4}[3\cos x + \cos 3x]$	$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

NOTE: A student may require formulae of class XI, other then above; therefore he is suggested to learn all trigonometric formulae studied in class XI.

(v). SOME SPECIAL INTEGRALS:

1. $\int \dfrac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\dfrac{x}{a}\right)+c$.	**1.** $\int \dfrac{1}{\sqrt{a^2-(bx+c)^2}}\,dx = \dfrac{1}{b}\sin^{-1}\left(\dfrac{bx+c}{a}\right)+c$.
2. $\int \dfrac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\dfrac{x}{a}\right)+c$.	**2.** $\int \dfrac{-1}{\sqrt{a^2-(bx+c)^2}}\,dx = \dfrac{1}{b}\cos^{-1}\left(\dfrac{bx+c}{a}\right)+c$.
3. $\int \dfrac{1}{a^2+x^2}\,dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)+c$.	**3.** $\int \dfrac{1}{a^2+(bx+c)^2}\,dx = \dfrac{1}{ab}\tan^{-1}\left(\dfrac{bx+c}{a}\right)+c$.
4. $\int \dfrac{-1}{a^2+x^2}\,dx = \dfrac{1}{a}\cot^{-1}\left(\dfrac{x}{a}\right)+c$	**4.** $\int \dfrac{-1}{a^2+(bx+c)^2}\,dx = \dfrac{1}{ab}\cot^{-1}\left(\dfrac{bx+c}{a}\right)+c$
5. $\int \dfrac{1}{x\sqrt{x^2-a^2}}\,dx = \dfrac{1}{a}\sec^{-1}\left(\dfrac{x}{a}\right)+c$.	**5.** $\int \dfrac{1}{(bx+c)\sqrt{(bx+c)^2-a^2}}\,dx = \dfrac{1}{ab}\sec^{-1}\left(\dfrac{bx+c}{a}\right)+c$.
6. $\int \dfrac{-1}{x\sqrt{x^2-a^2}}\,dx = \dfrac{1}{a}co\sec^{-1}\left(\dfrac{x}{a}\right)+c$.	**6.** $\int \dfrac{-1}{(bx+c)\sqrt{(bx+c)^2-a^2}}\,dx = \dfrac{1}{ab}co\sec^{-1}\left(\dfrac{bx+c}{a}\right)+c$.
7. $\int \dfrac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right|+c$.	**7.** $\int \dfrac{1}{\sqrt{(bx+c)^2-a^2}}\,dx = \dfrac{1}{b}\log\left|(bx+c)+\sqrt{(bx+c)^2-a^2}\right|+c$.
8. $\int \dfrac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right|+c$.	**8.** $\int \dfrac{1}{\sqrt{(bx+c)^2+a^2}}\,dx = \dfrac{1}{b}\log\left|(bx+c)+\sqrt{(bx+c)^2+a^2}\right|+c$.
9. $\int \dfrac{1}{x^2-a^2}\,dx = \dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right|+c$.	**9.** $\int \dfrac{1}{(bx+c)^2-a^2}\,dx = \dfrac{1}{2ab}\log\left|\dfrac{(bx+c)-a}{(bx+c)+a}\right|+c$.
10. $\int \dfrac{1}{a^2-x^2}\,dx = \dfrac{1}{2a}\log\left|\dfrac{a+x}{a-x}\right|+c$.	**10.** $\int \dfrac{1}{a^2-(bx+c)^2}\,dx = \dfrac{1}{2ab}\log\left|\dfrac{a+(bx+c)}{a-(bx+c)}\right|+c$.

3. INTEGRATION PARTIAL FRACTIONS:

FACTOR IN THE	CORRESPONDING PARTIAL FRACTION

DENOMIANTOR	
(Linear factor) $ax+b$	$\dfrac{A}{ax+b}$
Repeated linear factor **(i)** $(ax+b)^2$ **(ii)** $(ax+b)^n$	$\dfrac{A}{ax+b}+\dfrac{B}{(ax+b)^2}$ $\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\dfrac{A_3}{(ax+b)^3}+\ldots+\dfrac{A_n}{(ax+b)^n}$
Quadratic factor ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
Repeated quadratic factor **(i)** $(ax^2+bx+c)^2$ **(ii)** $(ax^2+bx+c)^n$	**(i)** $\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}$ **(ii)** $\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\dfrac{A_3x+B_3}{(ax^2+bx+c)^3}+\ldots+\dfrac{A_nx+B_n}{(ax^2+bx+c)^n}$

NOTE: Where A,B and A_i's and B_i's are real numbers and are to be calculated by an appropriate method

NOTE: If in an integration of the type $\dfrac{p(x)}{q(x)}$ **(i.e.) a rational expression** $\deg(p(x)) \geq \deg(q(x))$

then we first divide $p(x)$ **by** $q(x)$ **and write** $\dfrac{p(x)}{q(x)}$ **as**

$\dfrac{p(x)}{q(x)} = quotient + \dfrac{remainder}{divisor}$ **and then proceed.**

4. <u>INTEGRATION BY PARTS:</u>

Integration by parts is used in integrating functions of the type $f(x).g(x)$ **as follows.**

$$\int\left(I^{st}\ function \times II^{nd}\ function\right)dx = I^{st}\ function\int\left(II^{nd}\ function\right)dx - \int\left(\frac{d}{dx}\left(I^{st}\ function\right)\times\int\left(II^{nd}\ function\right)dx\right)dx$$

Where the I^{st} and II^{nd} functions are decided in the order of ILATE;

I: Inverse trigonometric function
L: Logarithmic function
T: Trigonometric functions
A: Algebraic functions
E: Exponential Functions

There are three type of questions based on integration by parts:

TYPE1. Directly based on the formulae

Example: $\int x\sin xdx$; $\int \log x dx$; $\int\left(\sin^{-1}x\right)^2 dx$ **etc.**

TYPE2: Integration of the type: $\int e^{ax}\sin bxdx$; $\int e^{ax}\cos bxdx$

TYPE3: Integration of the type:

$$\int e^x\left(f(x)+f'(x)\right)dx=e^x f(x)+c$$

$$\int e^{kx}\left(kf(x)+f'(x)\right)dx=e^{kx} f(x)+c$$

5. SOME MORE SPECIAL INTEGRALS

1. $\int\sqrt{a^2-x^2}dx=\frac{x\sqrt{x^2-a^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

2. $\int\sqrt{x^2+a^2}dx=\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c$

3. $\int\sqrt{x^2-a^2}dx=\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c$

NOTE: SOME MORE SPECIAL INTEGRALS OF THE TYPE f(ax+b)

1. $\int\sqrt{a^2-(bx+c)^2}dx=\frac{1}{\mathrm{b}}\left\{\frac{(bx+c)\sqrt{(bx+c)^2-a^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{bx+c}{a}\right)\right\}+c$

2. $\int\sqrt{(bx+c)^2+a^2}dx=\frac{1}{b}\left\{\frac{(bx+c)\sqrt{(bx+c)^2+a^2}}{2}+\frac{a^2}{2}\log\left|(bx+c)+\sqrt{(bx+c)^2+a^2}\right|\right\}+c$

3. $\int\sqrt{(bx+c)^2-a^2}dx=\frac{1}{b}\left\{\frac{(bx+c)\sqrt{(bx+c)^2-a^2}}{2}-\frac{a^2}{2}\log\left|(bx+c)+\sqrt{(bx+c)^2-a^2}\right|\right\}+c$

6. INTEGRATION OF THE TYPE: $\int\frac{x^2\pm1}{x^4+kx^2+1}dx$ **;** $\int\frac{1}{x^4+kx^2+1}dx$

METHOD:

STEP1: Divide the Nr. and Dr. by x^2. We get $1\pm\frac{1}{x^2}$ **in the Nr.**

STEP2: Introduce $\left(x\pm\frac{1}{x}\right)^2$ **in the Dr.**

STEP3: Put $x\pm\frac{1}{x}=t$ **, as per the situation and proceed.**

TYPES OF INTEGRATION OTHER THAN GIVEN IN THE N.C.E.R.T.

1. Integration of the type $\int \frac{1}{a+b\sin^2 x}dx$**,** $\int \frac{1}{a+b\cos^2 x}dx$**,** $\int \frac{1}{a\sin^2 x+b\cos^2 x}dx$

$\int \frac{1}{(a\sin x+b\cos x)^2}dx$

METHOD:

Step1. Divide Nr. and Dr. by $\sin^2 x$ **(or** $\cos^2 x$**)**

Step2. In the Dr. replace $\cos ec^2 x$ **by** $1+\cot^2 x$ **(or** $\sec^2 x$ **by** $1+\tan^2 x$**) and proceed.**

2. Integration of the type $\int \frac{1}{a+b\sin x}dx$**,** $\int \frac{1}{a+b\cos x}dx$**,** $\int \frac{1}{a\sin x+b\cos x}dx$ $\int \frac{1}{a\sin x+b\cos x+c}dx$

METHOD:

Step1. Replace $\sin x=\frac{2\tan x/2}{1+\tan^2 x/2}dx$ **and** $\cos x=\frac{1-\tan^2 x/2}{1+\tan^2 x/2}dx$

Step2. In the Nr. Replace $1+\tan^2 x/2=\sec^2 x/2$**.**

Step3. Put $\tan x/2=t$ **and proceed.**

3. Integration of the type.

TYPE:1. $\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}dx$

METHOD:

Put $a\sin x+b\cos x=\alpha\frac{d}{dx}(c\sin x+d\cos x)+\beta(c\sin x+d\cos x)$

Where α **and** β **are to be calculated by an appropriate method.**

TYPE:2. $\int \frac{a\sin x+b\cos x+c}{d\sin x+e\cos x+f}dx$

METHOD:

Put $a\sin x+b\cos x+c=\alpha\frac{d}{dx}(d\sin x+e\cos x+f)+\beta(d\sin x+e\cos x+f)+\gamma$

Where α **and** β **are to be calculated by an appropriate method.**

4. Integration of the type $\int \frac{\phi(x)}{P\sqrt{Q}}dx$**, where P and Q are either linear polynomial and quadratic polynomial alternately or simultaneously.**

CASE(i) If P & Q both are linear then put $Q=t^2$ **and proceed.**

CASE(ii) If P is quadratic & Q is linear then put $Q=t^2$ **and proceed.**

CASE(iii) If P is linear and Q is quadratic function of x, we put $P=\frac{1}{t}$**.**

CASE(iv) If P and Q both are pure quadratic of the form ax^2+b **then put** $x=\frac{1}{t}$**.**

Trigonometric Integrals

I. Integrating Powers of the Sine and Cosine Functions

A. Useful trigonometric identities

1. $\sin^2 x + \cos^2 x = 1$

2. $\sin 2x = 2\sin x \cos x$

3. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

4. $\sin^2 x = \dfrac{1 - \cos 2x}{2}$
5. $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
6. $\sin x \cos y = \dfrac{1}{2}[\sin(x-y) + \sin(x+y)]$
7. $\sin x \sin y = \dfrac{1}{2}[\cos(x-y) - \cos(x+y)]$
8. $\cos x \cos y = \dfrac{1}{2}[\cos(x-y) + \cos(x+y)]$

B. Reduction formulas

1. $\displaystyle\int \sin^n x\ dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\ dx$

2. $\displaystyle\int \cos^n x\ dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\ dx$

C. Examples

1. Find $\displaystyle\int \sin^2 x\ dx$.

Method 1(Integration by parts): $\displaystyle\int \sin^2 x\ dx = \int \sin x\ (\sin x\ dx)$. Let

$u = \sin x$ and $dv = \sin x\ dx \Rightarrow du = \cos x\ dx$ and $v = \displaystyle\int \sin x\, dx =$

$-\cos x$. Thus, $\displaystyle\int \sin^2 x\ dx = (\sin x)(-\cos x) + \int \cos^2 x\, dx = -\sin x \cos x +$

$\displaystyle\int (1 - \sin^2 x)\ dx = -\sin x \cos x + \int 1 dx - \int \sin^2 x\ dx = -\sin x \cos x + x -$

$$\int \sin^2 x\ dx \Rightarrow 2\int \sin^2 x\ dx = -\sin x \cos x + x \Rightarrow \int \sin^2 x\ dx =$$

$$-\frac{1}{2}\sin x \cos x + \frac{1}{2}x + C.$$

Method 2(Trig identity): $\int \sin^2 x\ dx = \frac{1}{2}\int (1-\cos 2x)\ dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$.

Method 3(Reduction formula): $\int \sin^2 x\ dx = -\frac{1}{2}\sin x \cos x + \frac{1}{2}\int 1 dx =$

$$-\frac{1}{2}\sin x \cos x + \frac{1}{2}x + C.$$

2. Find $\int \cos^3 x\ dx$.

Use the reduction formula: $\int \cos^3 x\ dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\int \cos x\, dx =$

$$\frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C = \frac{1}{3}\sin x(1-\sin^2 x) + \frac{2}{3}\sin x + C =$$

$$\sin x - \frac{1}{3}\sin^3 x + C.$$

3. Find $\int \sin^3 x \cos^2 x\ dx$.

$$\int \sin^3 x \cos^2 x\ dx = \int \sin^2 x \sin x \cos^2 x\ dx = \int (1-\cos^2 x)\cos^2 x \sin x dx =$$

$\int (\cos^2 x - \cos^4 x)(\sin x\ dx)$. Let $u = \cos x \Rightarrow du = -\sin x\, dx$. Thus,

$$\int (\cos^2 x - \cos^4 x)(\sin x\ dx) = -\int (u^2 - u^4)\ du = -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C =$$

$$-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C.$$

4. Find $\int \sin^2 x \cos^2 x\ dx$.

$$\int \sin^2 x \cos^2 x\ dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx = \frac{1}{4}\int (1-\cos^2 2x)\, dx =$$

$$\frac{1}{4}\int \sin^2 2x\, dx = \frac{1}{4}\int \left(\frac{1-\cos 4x}{2}\right) dx = \frac{1}{8}\int 1 dx - \frac{1}{8}\int \cos 4x\ dx =$$

$$\frac{1}{8}x - \frac{1}{32}\sin 4x + C.$$

5. Find $\int \sin 4x \cos 3x\ dx$.

Method 1(Integration by parts): Let $u = \sin 4x$ and $dv = \cos 3x\ dx \Rightarrow$ $du =$ $4\cos 4x\ dx$ and $v = \frac{1}{3}\sin 3x$. Thus, $\int \sin 4x \cos 3x\ dx =$ $(\sin 4x)\left(\frac{1}{3}\sin 3x\right) - \frac{4}{3}\int \cos 4x \sin 3x\ dx = \frac{1}{3}\sin 4x \sin 3x -$ $\frac{4}{3}\int \cos 4x \sin 3x\ dx$. Find $\int \cos 4x \sin 3x\ dx$. Let $u = \cos 4x$ and $dv =$ $\sin 3x\ dx \Rightarrow du = -4\sin 4x\ dx$ and $v = -\frac{1}{3}\cos 3x$. Thus, $\int \cos 4x \sin 3x\ dx = -\frac{1}{3}\cos 4x \cos 3x - \frac{4}{3}\int \sin 4x \cos 3x\ dx$. Returning to the original integral, $\int \sin 4x \cos 3x\ dx = \frac{1}{3}\sin 4x \sin 3x -$ $\frac{4}{3}\left\{-\frac{1}{3}\cos 4x \cos 3x - \frac{4}{3}\int \sin 4x \cos 3x\ dx\right\} = \frac{1}{3}\sin 4x \sin 3x +$ $\frac{4}{9}\cos 4x \cos 3x + \frac{16}{9}\int \sin 4x \cos 3x\ dx \Rightarrow -\frac{7}{9}\int \sin 4x \cos 3x\ dx =$ $\frac{1}{3}\sin 4x \sin 3x + \frac{4}{9}\cos 4x \cos 3x \Rightarrow \int \sin 4x \cos 3x\ dx =$ $-\frac{3}{7}\sin 4x \sin 3x - \frac{4}{7}\cos 4x \cos 3x + C$.

Method 2(Trig identity): $\int \sin 4x \cos 3x\ dx = \frac{1}{2}\int (\sin x + \sin 7x)\ dx =$ $-\frac{1}{2}\cos x - \frac{1}{14}\cos 7x + C$.

II. Integrating Powers of the Tangent and Secant Functions

A. Useful trigonometric identity: $\tan^2 x + 1 = \sec^2 x$

B. Useful integrals

1. $\int \sec x \tan x\ dx = \sec x + C$

2. $\int \sec^2 x\, dx = \tan x + C$

3. $\int \tan x\ dx = \ln|\sec x| + C = -\ln|\cos x| + C$

4. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

C. Reduction formulas

1. $\int \sec^n x\,dx = \dfrac{\sec^{n-2} x\tan x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2} x\,dx$

2. $\int \tan^n x\,dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

D. Examples

1. Find $\int \tan^2 x\,dx$.

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \int \sec^2 x\,dx - \int 1\,dx = \tan x - x + C.$$

2. Find $\int \tan^3 x dx$.

$$\int \tan^3 x dx = \frac{\tan^2 x}{2} - \int \tan x\,dx = \frac{1}{2}\tan^2 x - \ln|\sec x| + C.$$

3. Find $\int \sec^3 x dx$.

$$\int \sec^3 x\,dx = \frac{\sec x \tan x}{2} + \frac{1}{2}\int \sec x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

4. Find $\int \tan x \sec^2 x\,dx$.

Let $u = \tan x \Rightarrow du = \sec^2 x dx \Rightarrow \int \tan x \sec^2 x\,dx = \int u du = \frac{1}{2}u^2 + C =$
$\frac{1}{2}\tan^2 x + C$.

5. Find $\int \tan x \sec^4 x\,dx$.

$\int \tan x \sec^4 x\,dx = \int \tan x \sec^2 x \sec^2 x\,dx = \int \tan x(1+\tan^2 x)\sec^2 x\,dx =$
$\int \tan x \sec^2 x\,dx + \int \tan^3 x \sec^2 dx$. Let $u = \tan x \Rightarrow du = \sec^2 x\,dx$. Thus,
$\int \tan x \sec^4 x\,dx = \int u du + \int u^3 du = \frac{1}{2}u^2 + \frac{1}{4}u^4 + C = \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C$.

6. Find $\int \tan x \sec^3 x \, dx$.

$\int \tan x \sec^3 x \, dx = \int \sec^2 x (\sec x \tan x \, dx)$. Let $u = \sec x \Rightarrow du = \sec x \tan x dx$.

Thus, $\int \tan x \sec^3 x \, dx = \int u^2 du = \frac{1}{3} u^3 + C = \frac{1}{3} \sec^3 x + C$.

7. Find $\int \tan^2 x \sec^3 x \, dx$.

$\int \tan^2 x \sec^3 x \, dx = \int (\sec^2 x - 1) \sec^3 x \, dx = \int \sec^5 x \, dx - \int \sec^3 x \, dx$. Using the reduction formula, $\int \sec^5 x \, dx = \frac{1}{4} \sec^3 \tan x + \frac{3}{4} \int \sec^3 x \, dx$. Thus,

$\int \tan^2 x \sec^3 x \, dx = \int \sec^5 x \, dx - \int \sec^3 x \, dx = \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} \int \sec^3 x dx -$
$\int \sec^3 x \, dx = \frac{1}{4} \sec^3 x \tan x - \frac{1}{4} \int \sec^3 x \, dx = \frac{1}{4} \sec^3 x \tan x - \frac{1}{8} \sec x \tan x -$
$\frac{1}{8} \ln \left| \sec x + \tan x \right| + C$.

8. Find $\int \sqrt{\tan x} \; \sec^4 x \; dx$.

$\int \sqrt{\tan x} \; \sec^4 x \; dx = \int \sqrt{\tan x} \sec^2 x \sec^2 x \, dx = \int \sqrt{\tan x} (1 + \tan^2 x) \sec^2 x dx$.
Let $u = \tan x \Rightarrow du = \sec^2 x \, dx \Rightarrow \int \sqrt{\tan x} \; \sec^4 x \; dx = \int \sqrt{\tan x} \sec^2 x \, dx +$
$\int \sqrt{\tan x} \tan^2 x \sec^2 x \; dx = \int u^{1/2} du + \int u^{5/2} du = \frac{2}{3} u^{3/2} + \frac{2}{7} u^{7/2} + C =$
$\frac{2}{3} (\tan x)^{3/2} + \frac{2}{7} (\tan x)^{7/2} + C$.

9. Find $\int \sqrt{\sec x} \tan x \, dx$.

Let $u = \sqrt{\sec x} \Rightarrow u^2 = \sec x \Rightarrow 2u du = \sec x \tan x dx = u^2 \tan x dx \Rightarrow$
$\tan x \, dx = \frac{2u du}{u^2} = \frac{2}{u} du$. Thus, $\int \sqrt{\sec x} \tan x \, dx = \int u \left(\frac{2}{u} du \right) = 2 \int 1 du =$
$2u + C = 2\sqrt{\sec x} + C$.

Practice Sheet forTrigonometric Integrals

(1) Prove the reduction formula: $\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

(2) Prove the reduction formula: $\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$

(3) Prove the reduction formula: $\int \sec^n x\,dx = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$

(4) Prove the reduction formula: $\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

(5) $\int_0^{\pi/4} \tan^3(3x)\ dx =$

(6) $\int_0^{\pi/4} \cos^2(2x)\ dx =$

(7) $\int_0^{\pi/8} \sin(5x)\cos(3x)\ dx =$

(8) $\int \tan^3 x\sec^3 x\ dx =$

(9) $\int \sqrt{\sin x}\cos^3 x\ dx =$

(10) $\int \cos^3 x\sin^2 x\ dx =$

(11) $\int_0^{\pi/2} \frac{\sin^3 x}{\sqrt{\cos x}}\ dx =$

(12) $\int \sin^2 x\cos^2 x dx =$

(13) $\int \tan^5 x \sec x dx =$

<u>Solution Key for Trigonometric Integrals</u>

(1) $\int \sin^n x\,dx = \int \sin^{n-1} x \sin x\,dx$. Use integration by parts with $u = \sin^{n-1} x$ and

$dv = \sin x\,dx \Rightarrow du = (n-1)\sin^{n-2} x \cos x\,dx$ and $v = \int \sin x\,dx = -\cos x \Rightarrow$

$\int \sin^n x\,dx = \int \sin^{n-1} x \sin x\,dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx =$

$-\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\left(1-\sin^2 x\right)dx = -\sin^{n-1} x \cos x +$

$(n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx \Rightarrow n\int \sin^n x\,dx = -\sin^{n-1} x \cos x +$

$(n-1)\int \sin^{n-2} x\,dx \Rightarrow \int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$

(2) $\int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx$. Use integration by parts with $u = \cos^{n-1} x$ and

$dv = \cos x\,dx \Rightarrow du = (n-1)\cos^{n-2} x(-\sin x)\,dx$ and $v = \int \cos x\,dx = \sin x \Rightarrow$

$\int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx =$

$\cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\left(1-\cos^2 x\right)dx = \cos^{n-1} x \sin x +$

$(n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx \Rightarrow n\int \cos^n x\,dx = \cos^{n-1} x \sin x +$

$(n-1)\int \cos^{n-2} x\,dx \Rightarrow \int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$

(3) $\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx$. Use integration by parts with $u = \sec^{n-2} x$ and

$dv = \sec^2 x\,dx \Rightarrow du = (n-2)\sec^{n-3} x(\sec x \tan x\,dx)$ and $v = \int \sec^2 x\,dx = \tan x \Rightarrow$

$$\int \sec^n x\,dx = \int \sec^{n-2} x \sec^2 x\,dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,dx =$$

$$\sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\left(\sec^2 x - 1\right)dx = \sec^{n-2} x \tan x - (n-2)\int \sec^n x dx +$$

$$(n-2)\int \sec^{n-2} x\,dx \Rightarrow (n-1)\int \sec^n x\,dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\,dx \Rightarrow$$

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx.$$

(4) $\int \tan^n x\,dx = \int \tan^{n-2} x \tan^2 x\,dx = \int \tan^{n-2} x\left(\sec^2 x - 1\right)dx = \int \tan^{n-2} x \sec^2 x dx -$

$$\int \tan^{n-2} x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx.$$

(5) Let $u = 3x \Rightarrow du = 3\,dx \Rightarrow \int \tan^3(3x)\,dx = \frac{1}{3}\int \tan^3(3x)\;3dx = \frac{1}{3}\int \tan^3 u\,du$. Use reduction formula #4 above to get $\frac{1}{3}\int \tan^3 u\,du = \frac{1}{3}\left(\frac{\tan^2 u}{2}\right) - \frac{1}{3}\int \tan u\,du =$

$$\frac{1}{6}\tan^2 u - \frac{1}{3}\ln|\sec u| \Rightarrow \int_0^{\pi/4} \tan^3(3x)\;\;dx = \left\{\frac{1}{6}\tan^2(3x) - \frac{1}{3}\ln|\sec(3x)|\right\}_0^{\pi/4} =$$

$$\left\{\frac{1}{6}\tan^2\left(\frac{3\pi}{4}\right) - \frac{1}{3}\ln\left|\sec\left(\frac{3\pi}{4}\right)\right|\right\} - \left\{\frac{1}{6}\tan^2(0) - \frac{1}{3}\ln|\sec(0)|\right\} = \frac{1}{6}(-1)^2 - \frac{1}{3}\ln\left|-\sqrt{2}\right| -$$

$$\frac{1}{6}(0)^2 + \frac{1}{3}\ln 1 = \frac{1}{6} - \frac{1}{3}\ln\left(\sqrt{2}\right).$$

(6) Use the trigonometric identity $\cos^2\Delta = \frac{1+\cos 2\Delta}{2}$ to get $\int \cos^2(2x)\,dx =$

$$\int \frac{1+\cos(4x)}{2}\,dx = \frac{1}{2}\int 1dx + \frac{1}{2}\int \cos(4x)\,dx = \frac{1}{2}x + \frac{1}{8}\sin(4x) \Rightarrow \int_0^{\pi/4} \cos^2(2x)\;dx =$$

$$\left\{\frac{1}{2}\left(\frac{\pi}{4}\right) + \frac{1}{8}\sin\pi\right\} - \left\{\frac{1}{2}(0) + \frac{1}{8}\sin(0) = \frac{\pi}{8}\right\}.$$

(7) Use the trigonometric identity $\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)]$ to get

$$\int \sin(5x)\cos(3x)\,dx = \frac{1}{2}\int \sin(2x)\,dx + \frac{1}{2}\int \sin(8x)\,dx = -\frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x) \Rightarrow$$

$$\int_0^{\pi/8} \sin(5x)\cos(3x)\,dx = \left\{-\frac{1}{4}\cos\left(\frac{\pi}{4}\right) - \frac{1}{16}\cos(\pi)\right\} - \left\{-\frac{1}{4}\cos 0 - \frac{1}{16}\cos 0\right\} =$$

$$-\frac{1}{4}\left(\frac{\sqrt{2}}{2}\right) + \frac{1}{16} + \frac{1}{4} + \frac{1}{16} = \frac{3-\sqrt{2}}{8}$$

(8) $\int \tan^3 x \sec^3 x\;dx = \int \tan^2 x \sec^2 x\,(\sec x \tan x\,dx) =$

$$\int (\sec^2 x - 1)\sec^2 x\,(\sec x \tan x\,dx) = \int \sec^4 x\,(\sec x \tan x\,dx) -$$

$$\int \sec^2 x\,(\sec x \tan x\,dx) = \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C.$$

(9) $\int \sqrt{\sin x}\cos^3 x\;dx = \int \sqrt{\sin x}\;(\cos^2 x)(\cos x\,dx) = \int (\sin x)^{1/2}\left(1-\sin^2 x\right)\cos x dx =$

$$\int (\sin x)^{1/2}\cos x\,dx - \int (\sin x)^{5/2}\cos x\,dx = \frac{2}{3}(\sin x)^{3/2} - \frac{2}{7}(\sin x)^{7/2} + C.$$

(10) $\int \cos^3 x \sin^2 x\;dx = \int \cos^2 x \sin^2 x\left(\cos x\,dx\right) = \int \left(1-\sin^2 x\right)\left(\sin^2 x\right)\cos x\,dx =$

$$\int \sin^2 x(\cos x\,dx) - \int \sin^4 x(\cos x\,dx) = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C.$$

(11) $\int \frac{\sin^3 x}{\sqrt{\cos x}}\;dx = \int (\cos x)^{-1/2}\sin^2 x\,(\sin x\,dx = \int (\cos x)^{-1/2}\left(1-\cos^2 x\right)\sin x\,dx =$

$$\int (\cos x)^{-1/2}(\sin x\,dx) - \int (\cos x)^{3/2}(\sin x\,dx) = -2(\cos x)^{1/2} + \frac{2}{5}(\cos x)^{5/2} \Rightarrow$$

$$\int_0^{\pi/2} \frac{\sin^3 x}{\sqrt{\cos x}}\,dx = \left\{-2\cos\left(\frac{\pi}{2}\right) + \frac{2}{5}\left(\cos\left(\frac{\pi}{2}\right)\right)^{5/2}\right\} - \left\{-2\cos 0 + \frac{2}{5}(\cos 0)^{5/2}\right\} = \frac{8}{5}.$$

(12) Use the trigonometric identities $\cos^2\Delta = \frac{1+\cos 2\Delta}{2}$ and $\sin^2\Delta = \frac{1-\cos 2\Delta}{2}$.

$$\int \sin^2 x\cos^2 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{4}\int (1-\cos^2 2x)\,dx =$$

$$\frac{1}{4}\int 1\,dx - \frac{1}{4}\int \cos^2 2x\,dx = \frac{1}{4}x - \frac{1}{4}\int\left(\frac{1+\cos 4x}{2}\right)dx = \frac{1}{4}x - \frac{1}{8}\int 1\,dx -$$

$$\frac{1}{8}\int \cos 4x\,dx = \frac{1}{4}x - \frac{1}{8}x - \frac{1}{32}\sin 4x + C = \frac{1}{8}x - \frac{1}{32}\sin 4x + C.$$

(13) $\int \tan^5 x\sec x\,dx = \int \tan^4 x\tan x\sec x\,dx = \int (\tan^2 x)^2 \tan x\sec x\,dx =$

$$\int (\sec^2 x - 1)^2 \sec x\tan x\,dx = \int (\sec^4 x - 2\sec^2 x + 1)\sec x\tan x\,dx =$$

$$\int \sec^4 x(\sec x\tan x\,dx) - 2\int \sec^2 x(\sec x\tan x\,dx) + \int \sec x\tan x\,dx =$$

$$\frac{1}{5}\sec^5 x - \frac{2}{3}\sec^3 x + \sec x + C.$$

UNIT - V
VECTOR ALGEBRA

Contents

Vectors and Scalars

A vector is a quantity that has **size** (magnitude) and **direction**. Examples of vectors are velocity, acceleration, force, displacement and moment. A force 10N upwards is a vector.

So what are scalars?

A scalar is a quantity that has size but **no** direction. Examples of scalars are mass, length, time, volume, speed and temperature.

How do we write down vectors and scalars and how can we distinguish between them?

A vector **from** O **to** A is denoted by $\overrightarrow{OA}$ or written in bold typeface **a** and can be represented geometrically as:

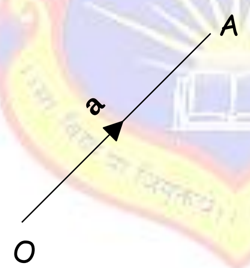


Fig 1

A scalar is denoted by a , not in bold, so that we can distinguish between vectors and scalars.
Two vectors are equivalent if they have the **same** direction and magnitude. For example the vectors **d** and **e** in Fig 2 are equivalent.

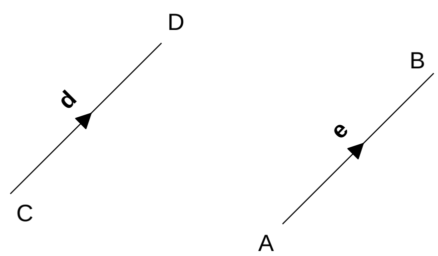


Fig 2

The vectors **d** and **e** have the same direction and magnitude but only differ in position. Also note that the direction of the arrow gives the direction of the vector, that is $\overrightarrow{CD}$ is different from $\overrightarrow{DC}$.

The magnitude or length of the vector $\overrightarrow{AB}$ is denoted by $\left|\overrightarrow{AB}\right|$.

There are many examples of vectors in the real world:

(a) A displacement of 20m to the horizontal right of an object from O to A:

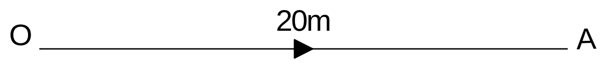


Fig 3

(b) A force on an object acting vertically downwards:

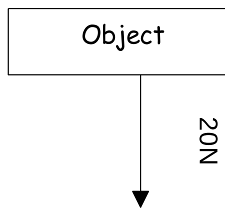


Fig 4

(c) The velocity and acceleration of a particle thrown vertically upwards:

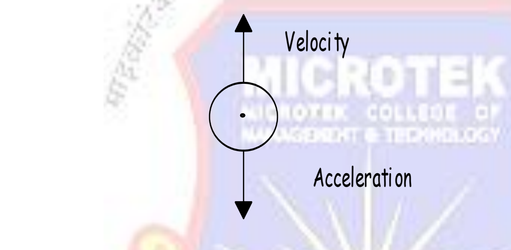


Fig 5

A2 Vector Addition and Scalar Multiplication

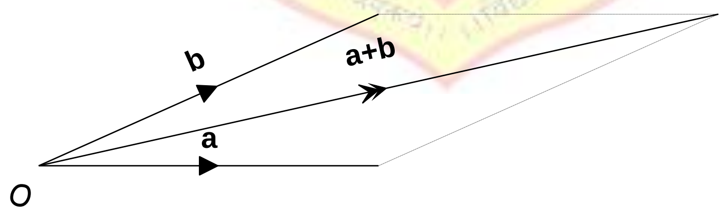


Fig 6

The result of adding two vectors such as $\mathbf{a}$ and $\mathbf{b}$ in Fig 6 is the diagonal of the parallelogram, $\mathbf{a}+\mathbf{b}$, as shown in Fig 6.

The multiplication $k\mathbf{a}$ of a real number k with a vector $\mathbf{a}$ is the product of the size of $\mathbf{a}$ with the number k. For example $2\mathbf{a}$ is the vector in the same direction as vector $\mathbf{a}$ but the magnitude is twice as long.

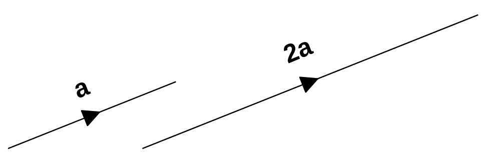


Fig 7

What does the vector $\frac{1}{2}\mathbf{a}$ *look like?*

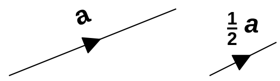


Fig 8

Same direction as vector **a** but half the magnitude.

What effect does a negative k have on a vector such as $k\mathbf{a}$ *?*

If $k = -2$ then $-2\mathbf{a}$ is the vector **a** but in the opposite direction and the magnitude is multiplied by 2, that is:

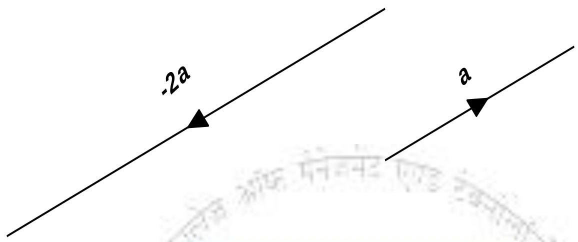


Fig 9

A vector $-\mathbf{a}$ is the vector **a** but in the opposite direction. We can define this as

$$-\mathbf{a} = (-1)\mathbf{a}$$

We call the product $k\mathbf{a}$ scalar multiplication.

We can also subtract vectors as the next diagram shows:

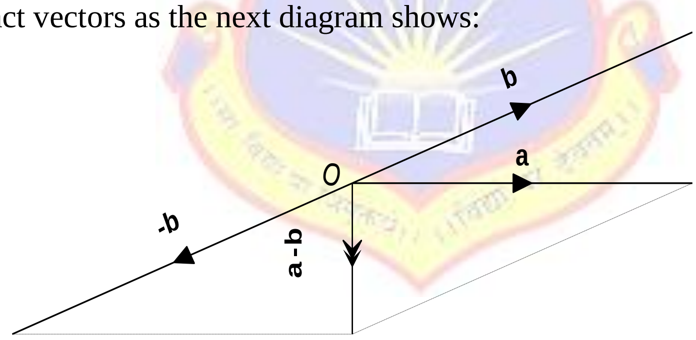


Fig 10

The vector subtraction of two vectors **a** and **b** is defined by

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

A3 Vectors in $\mathbb{R}^2$

What is meant by $\mathbb{R}^2$ *?*

$\mathbb{R}^2$ is the plane representing the Cartesian coordinate system named after the French mathematician (philosopher) Rene Descartes.

Rene Descartes was a French philosopher born in 1596. He attended a Jesuit college and because of his poor health he was allowed to remain in bed until 11 o'clock in the morning, a habit he continued until his death in 1650.
Descartes studied law at the University of Poitiers which is located south west of Paris. After graduating in 1618 he went to Holland to study mathematics.
Over the next decade he travelled through Europe eventually

Fig 11 Rene Descartes 1596 to 1650

Descartes main contribution to mathematics was his analytic geometry which included our present *x-y* plane and the three dimensional space.
In 1649 Descartes moved to Sweden to teach Queen Christina. However she wanted to learn her mathematics early in the morning (5am) which did not suit Descartes because he had a habit of getting up at 11am. Combined with these 5am starts and the harsh Swedish winter Descartes died of pneumonia in 1650.
The points in the plane are ordered pairs with reference to the origin which is denoted by O. For example the following are all vectors in the plane $\mathbb{R}^2$:

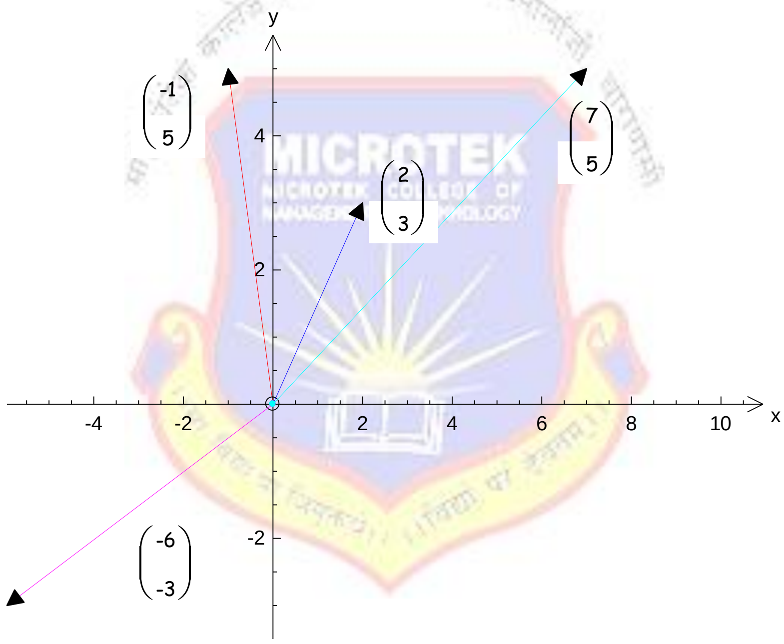


Fig 12

These are examples of vectors with two entries, $\begin{pmatrix}-6\\-3\end{pmatrix}$, $\begin{pmatrix}7\\5\end{pmatrix}$, $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}-1\\5\end{pmatrix}$.

The set of **all** vectors with two entries is denoted by $\mathbb{R}^2$ and pronounced "r two". The $\mathbb{R}$ represents that the entries are real numbers.
We add and subtract vectors in $\mathbb{R}^2$ as stated above, that is we apply the parallelogram law on the vectors. For example:

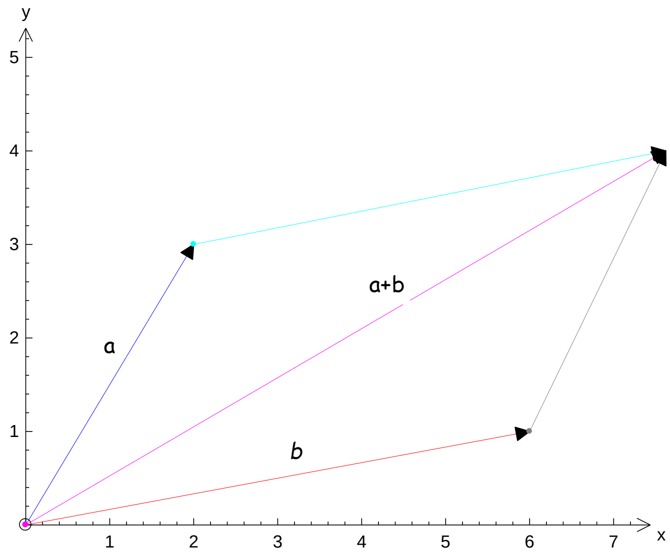


Fig 13

What does the term ordered pair mean?

The order of the entries matters, that is the coordinate $(a,\ b)$ is **different** from $(b,\ a)$ provided $a \neq b$.

Normally the coordinate $(a,\ b)$ is written as a column vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

Example 1

Let $\mathbf{u} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$. Plot $\mathbf{u}+\mathbf{v}$ and write down $\mathbf{u}+\mathbf{v}$ as a column vector. *What do you notice about your result?*

Solution

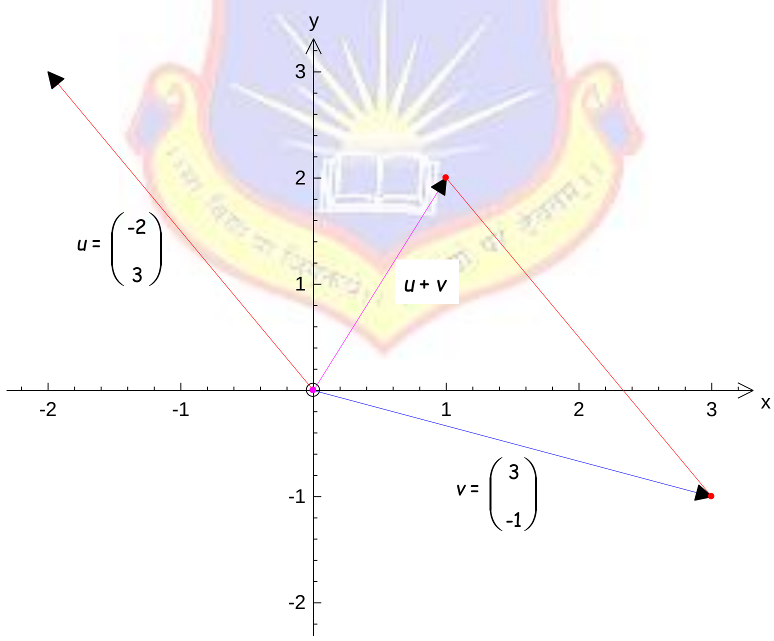


Fig 14

By examining Fig 14 we have that the coordinates of $\mathbf{u}+\mathbf{v}$ are $(1,\ 2)$ and this is written as a column vector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

If we add x and y coordinates separately then we obtain the resultant vector.

That is if we evaluate $\mathbf{u}+\mathbf{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3-2 \\ -1+3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ which means that we can add the corresponding entries of the vector to find $\mathbf{u}+\mathbf{v}$.

In general if $\mathbf{u}=\begin{pmatrix}a\\b\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}c\\d\end{pmatrix}$ then

$$\mathbf{u}+\mathbf{v}=\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}a+c\\b+d\end{pmatrix}$$

Example 2

Let $\mathbf{v}=\begin{pmatrix}3\\1\end{pmatrix}$. Plot the vectors $\frac{1}{2}\mathbf{v}$, $2\mathbf{v}$, $3\mathbf{v}$ and $-\mathbf{v}$ on the same axes.

Solution. Plotting each of these vectors on $\mathbb{R}^2$ we have

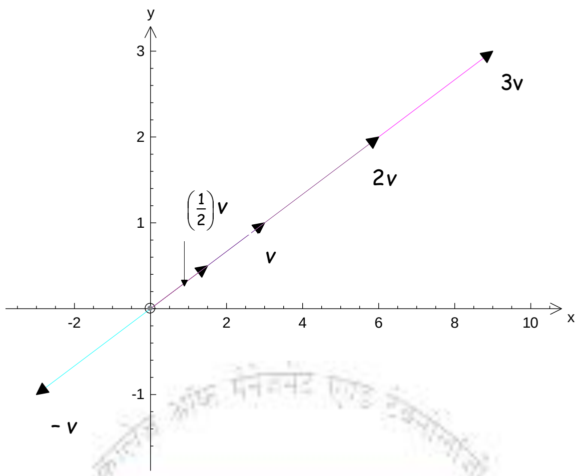


Fig 15

Note that by reading off the coordinates of each vector we have:

$$\frac{1}{2}\mathbf{v}=\frac{1}{2}\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}1.5\\0.5\end{pmatrix},\ 2\mathbf{v}=2\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}6\\2\end{pmatrix},\ 3\mathbf{v}=3\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}9\\3\end{pmatrix}\text{ and }-\mathbf{v}=-\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}-3\\-1\end{pmatrix}$$

Remember the product $k\mathbf{v}$ is called scalar multiplication. The term **scalar** comes from the Latin word **scala** meaning ladder. Scalar multiplication changes the length of the vector or we can say it changes the scale of the vector as you can see in Fig 15.

In general if $\mathbf{v}=\begin{pmatrix}a\\b\end{pmatrix}$ then the scalar multiplication

$$k\mathbf{v}=k\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}ka\\kb\end{pmatrix}$$

A4 Vectors in $\mathbb{R}^3$

What does the notation $\mathbb{R}^3$ mean?

$\mathbb{R}^3$ is the set of all ordered triples of real numbers and is also called 3-space.

We can extend the vector properties in $\mathbb{R}^2$ mentioned in subsection A3 above to three dimensions $\mathbb{R}^3$ pronounced "r three".

The $x-y$ plane can be extended to cover three dimensions by including a third axis called the *z* axis. This axes is at right angles to the other two, *x* and *y*, axes. The position of a vector in three dimensions is given by three co-ordinates $(x,\ y,\ z)$.

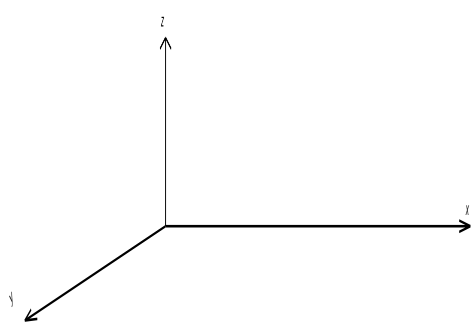
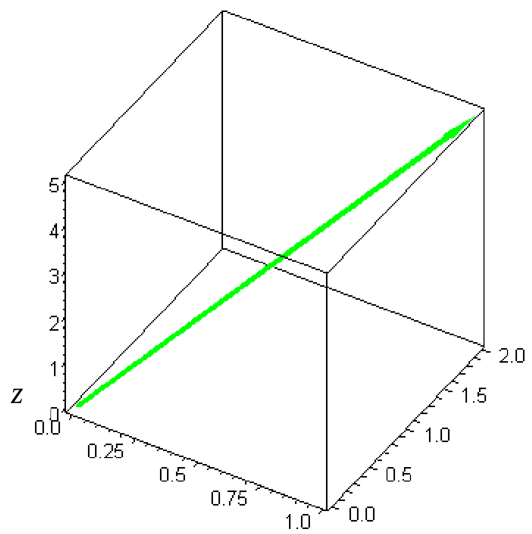


Fig 16

Shows the 3 axes x, y and z.

For example the following is the vector $\begin{pmatrix}1\\2\\5\end{pmatrix}$ in $\mathbb{R}^3$ and is represented geometrically by:



$$\begin{pmatrix}1\\2\\5\end{pmatrix}$$

Fig 17

Vector addition and scalar multiplication is carried out as in the plane $\mathbb{R}^2$. That is if

$\mathbf{u} = \begin{pmatrix}a\\b\\c\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}d\\e\\f\end{pmatrix}$ then the vector addition

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix}a\\b\\c\end{pmatrix} + \begin{pmatrix}d\\e\\f\end{pmatrix} = \begin{pmatrix}a+d\\b+e\\c+f\end{pmatrix}$$

Scalar multiplication is defined by

$$k\mathbf{u} = k\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}ka\\kb\\kc\end{pmatrix}$$

A5 Vectors in $\mathbb{R}^n$

What does $\mathbb{R}^n$ represent?

In the 17th century Rene Descartes used ordered pairs of real numbers, $\mathbf{v} = \begin{pmatrix}a\\b\end{pmatrix}$, to describe vectors in the plane and extended it to ordered triples of real numbers,

$\mathbf{v} = \begin{pmatrix}a\\b\\c\end{pmatrix}$, to describe vectors in 3 dimensional space. *Why can't we extend this to an* *ordered quadruple of real numbers,* $\mathbf{v}=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$*, or n- tuples of real numbers,* $\mathbf{v}=\begin{pmatrix}v_1\\v_2\\\vdots\\v_n\end{pmatrix}$*?*

In the 17th century vectors were defined as geometric objects and there was **no** geometric interpretation of $\mathbb{R}^n$ for n greater than 3. However in the 19th century vectors were thought of as mathematical objects that can be added, subtracted, scalar multiplied etc so we could extend the vector definition. An example is a system of linear equations where the number of unknowns $x_1,\ x_2,\ x_3,\ \cdots$ and x_n is greater than 3.

A vector $\mathbf{v}=\begin{pmatrix}v_1\\v_2\\\vdots\\v_n\end{pmatrix}$ is called an *n* dimensional vector. An example is $\mathbf{v}=\begin{pmatrix}1\\-2\\\vdots\\8\end{pmatrix}$.

Hence $\mathbb{R}^n$ is the set of all *n* dimensional vectors where $\mathbb{R}$ signifies that the entries of the vector are real numbers, that is $v_1,\ v_2,\ v_3,\ \cdots$ and v_n are **all** real numbers. The real number v_j of the vector **v** is called the **component** or more precisely the jth component of the vector **v**.

This $\mathbb{R}^n$ is also called *n*-space or the vector space of *n*-tuples.

Note that the vectors are ordered *n*-tuples. *What does this mean?*

The vector $\mathbf{v}=\begin{pmatrix}1\\-2\\\vdots\\8\end{pmatrix}$ is **different** from $\begin{pmatrix}-2\\1\\\vdots\\8\end{pmatrix}$, that is the order of the components matters.

How do we draw vectors in $\mathbb{R}^n$ *for* $n\geq 4$*?*

We **cannot** draw pictures of vectors in $\mathbb{R}^4,\ \mathbb{R}^5,\ \mathbb{R}^6$ etc. *What is the point of the n-space,* $\mathbb{R}^n$*, for* $n\geq 4$*?*

Well we can carry out vector arithmetic in *n*-space.

A6 Vector Addition and Scalar Multiplication in $\mathbb{R}^n$

Geometric interpretation of vectors in $\mathbb{R}^n$ is **not** possible for $n\geq 4$ therefore we define vector addition and scalar multiplication by algebraic means.

Two vectors **u** and **v** are **equal** if they have the same number of components and the corresponding components are equal. *How can we write this in mathematical notation?*

Let $\mathbf{u}=\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}v_1\\v_2\\\vdots\\v_n\end{pmatrix}$ and if

(3.2) $u_j=v_j$ for $j=1,\ 2,\ 3,\ \cdots,\ n$ then the vectors $\mathbf{u}=\mathbf{v}$.

For example the vectors $\begin{pmatrix}1\\5\\7\end{pmatrix}$ and $\begin{pmatrix}1\\7\\5\end{pmatrix}$ are **not** equal because the corresponding components are **not** equal.

Example 3

Let $\mathbf{u} = \begin{pmatrix} x-3 \\ y+1 \\ z+x \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. If $\mathbf{u} = \mathbf{v}$ then determine the real numbers $x,\ y$ and z.

Solution.

Since $\mathbf{u} = \mathbf{v}$ we have

$$\begin{aligned} x-3=1 &\quad \text{gives } x=4 \\ y+1=2 &\quad \text{gives } y=1 \\ z+x=3 &\quad \text{gives } z+4=3 \ \Rightarrow \ z=-1 \end{aligned}$$

Our solution is $x=4,\ y=1$ and $z=-1$.

We can also define vector addition and scalar multiplication in $\mathbb{R}^n$.

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ be vectors in $\mathbb{R}^n$ then

(3.3) $$\mathbf{u}+\mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{pmatrix}$$

The sum of the vectors **u** and **v** denoted by $\mathbf{u}+\mathbf{v}$ is executed by adding the corresponding components as formulated in (3.3). Note that $\mathbf{u}+\mathbf{v}$ is also a vector in $\mathbb{R}^n$.

Scalar multiplication $k\mathbf{v}$ is carried out by multiplying each component of the vector **v** by the real number *k*:

(3.4) $$k\mathbf{v} = k\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{pmatrix}$$

Again $k\mathbf{v}$ is a vector in $\mathbb{R}^n$.

Example 4

Let $\mathbf{u} = \begin{pmatrix} -3 \\ 1 \\ 7 \\ -5 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 9 \\ 2 \\ -4 \\ 1 \end{pmatrix}$. Find

(a) $\mathbf{u}+\mathbf{v}$ (b) $10\mathbf{u}$ (c) $3\mathbf{u}+2\mathbf{v}$ (d) $\mathbf{u}-\mathbf{u}$ (e) $-2\mathbf{u}-8\mathbf{v}$

Solution.

(a) By applying (3.3) we have

$$\mathbf{u}+\mathbf{v} = \begin{pmatrix} -3 \\ 1 \\ 7 \\ -5 \end{pmatrix} + \begin{pmatrix} 9 \\ 2 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -3+9 \\ 1+2 \\ 7-4 \\ -5+1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 3 \\ -4 \end{pmatrix}$$

(b) By using (3.4) we have

$$10\mathbf{u} = 10\begin{pmatrix} -3 \\ 1 \\ 7 \\ -5 \end{pmatrix} = \begin{pmatrix} -3\times 10 \\ 1\times 10 \\ 7\times 10 \\ -5\times 10 \end{pmatrix} = \begin{pmatrix} -30 \\ 10 \\ 70 \\ -50 \end{pmatrix}$$

(c) By applying both (3.3) and (3.4) we have

$$3\mathbf{u} + 2\mathbf{v} = 3\begin{pmatrix} -3 \\ 1 \\ 7 \\ -5 \end{pmatrix} + 2\begin{pmatrix} 9 \\ 2 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -3\times 3 \\ 1\times 3 \\ 7\times 3 \\ -5\times 3 \end{pmatrix} + \begin{pmatrix} 9\times 2 \\ 2\times 2 \\ -4\times 2 \\ 1\times 2 \end{pmatrix}$$

$$= \begin{pmatrix} -9 \\ 3 \\ 21 \\ -15 \end{pmatrix} + \begin{pmatrix} 18 \\ 4 \\ -8 \\ 2 \end{pmatrix} = \begin{pmatrix} -9+18 \\ 3+4 \\ 21-8 \\ -15+2 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \\ 13 \\ -13 \end{pmatrix}$$

(d) We have

$$\mathbf{u} - \mathbf{u} = \begin{pmatrix} -3 \\ 1 \\ 7 \\ -5 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 7 \\ -5 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 7 \\ -5 \end{pmatrix} + (-1)\begin{pmatrix} -3 \\ 1 \\ 7 \\ -5 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 1 \\ 7 \\ -5 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \\ -7 \\ 5 \end{pmatrix} = \begin{pmatrix} -3+3 \\ 1-1 \\ 7-7 \\ -5+5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{O}$$

Hence $\mathbf{u} - \mathbf{u}$ gives the zero vector **O**.

(e) We have

$$-2\mathbf{u} - 8\mathbf{v} = -2\begin{pmatrix} -3 \\ 1 \\ 7 \\ -5 \end{pmatrix} - 8\begin{pmatrix} 9 \\ 2 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -3\times(-2) \\ 1\times(-2) \\ 7\times(-2) \\ -5\times(-2) \end{pmatrix} - \begin{pmatrix} 9\times 8 \\ 2\times 8 \\ -4\times 8 \\ 1\times 8 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -2 \\ -14 \\ 10 \end{pmatrix} - \begin{pmatrix} 72 \\ 16 \\ -32 \\ 8 \end{pmatrix} = \begin{pmatrix} 6-72 \\ -2-16 \\ -14-(-32) \\ 10-8 \end{pmatrix} = \begin{pmatrix} -66 \\ -18 \\ 18 \\ 2 \end{pmatrix}$$

You may like to check these results of Example 4 in MATLAB.

Note that for any vector **v** we have

$$\mathbf{v} - \mathbf{v} = \mathbf{O}$$

The zero vector in $\mathbb{R}^n$ is denoted by **O** and is defined as

(3.5) $$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$ [All entries are zero]

There are other algebraic properties of vectors which we describe in the next section.

Why is this chapter called Euclidean Space?

Euclidean space is the space of **all** *n*-tuples of real numbers which is denoted by $\mathbb{R}^n$.

Hence Euclidean space is the set $\mathbb{R}^n$.

Euclid was a Greek mathematician who lived around 300BC and developed distances and angles in the plane and three dimension space. A more detailed profile of Euclid is given in the next section.

SUMMARY

Vectors have magnitude as well direction. Scalars only have magnitude. Vectors are normally denoted by bold letters such as **u**, **v**, **w** etc.

Vector addition in the plane $\mathbb{R}^2$ is carried out by the parallelogram rule and scalar multiplication scales the vector according to the multiple *k*.

$\mathbb{R}^2$ is also called 2-space.

$\mathbb{R}^3$ is the three dimensional space with x, y and *z* axis at right hand angles to each other. $\mathbb{R}^3$ is also called 3-space.

We can extend the above space to *n*-space which is denoted by $\mathbb{R}^n$ where *n* is a natural number such as 1, 2, 3, 4 ,5 …

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ be vectors in $\mathbb{R}^n$ then

(3.3) $$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}$$

(3.4) $$k\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} kv_1 \\ kv_2 \\ \vdots \\ kv_n \end{pmatrix}$$

<u>Double and Triple Scalar and Vector Product and physical interpretation of area and volume</u>

Fundamental Concepts

1. Scalar quantities: mass, density, area, time, potential, temperature, speed, work, etc.
2. Vectors are physical quantities which have the property of directions and magnitude.
   e.g. Velocity $\vec{v}$, weight $\vec{w}$, force $\vec{f}$, etc.
3. Properties:

   (a) The magnitude of $\vec{u}$ is denoted by $|\vec{u}|$.

   (b) $\overrightarrow{AB} = \overrightarrow{CD}$ if and only if $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{CD}\right|$, and $\overrightarrow{AB}$ and $\overrightarrow{CD}$ has the same direction.

   (c) $\overrightarrow{AB} = -\overrightarrow{BA}$

   (d) Null vector, zero vector $\vec{0}$, is a vector with zero magnitude i.e. $|\vec{0}| = 0$.

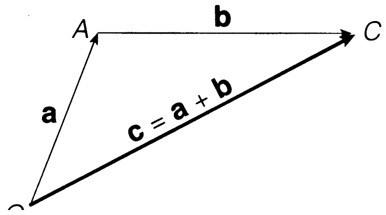
   The direction of a zero vector is indetermine.

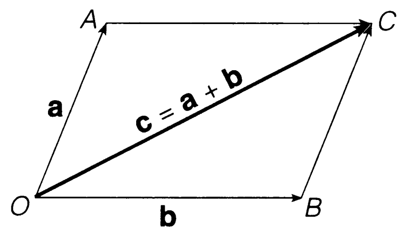
   (e) Unit vector, $\hat{u}$ or $\overrightarrow{e_u}$, is a vector with magnitude of 1 unit. I.e. $|\vec{u}| = 1$.

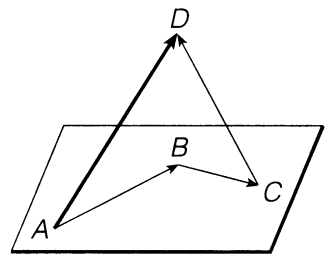
(f) $\hat{u} = \dfrac{\vec{u}}{|\vec{u}|} \Leftrightarrow \vec{u} = |\vec{u}|\hat{u}$

7.2 Addition and Subtraction of Vectors

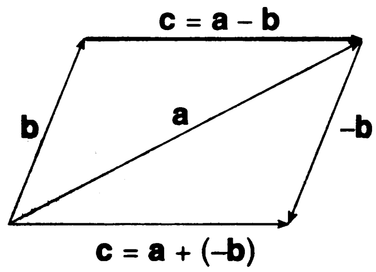
1. Geometric meaning of addition and subtraction.

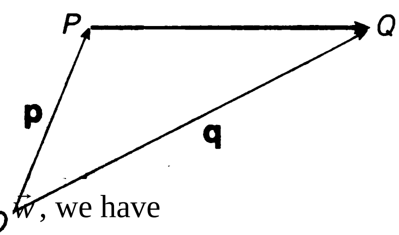






$\overline{AB} + \overline{BC} + \overline{CD} = \overline{AD}$





$\overrightarrow{PQ} = \boldsymbol{q} - \boldsymbol{p}$

2. $\vec{v}$ and $\vec{w}$, we have

$\vec{v} = \vec{v} + \vec{u}$,

(b) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$,

(c) $\vec{u} + \vec{0} = \vec{0} + \vec{u}$

(d) $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$

N.B. (1) $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$

(2) $\vec{c} = \vec{a} + \vec{b} \Rightarrow \vec{a} = \vec{c} - \vec{b}$

7.3 Scalar Multiplication

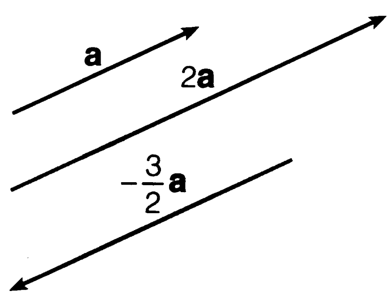
When a vector a is multiplied by a scalar m, the product ma is a vector parallel to a such that

(a) The magnitude of ma is $|m|$ times that of a.

(b) When $m > 0$, ma has the same direction as that of a,

When $m < 0$, ma has the opposite direction as that of a.

These properties are illustrated in Figure.



Theorem **Properties of Scalar Multiplication**

Let m, n be two scalars. For any two vectors a and b, we have

(a) $m(na) = (mn)a$

(b) $(m+n)a = ma + na$

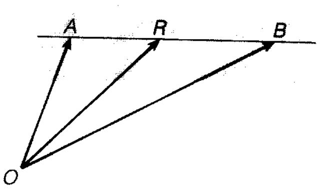
(c) $m(a+b) = ma + mb$

(d) $1a = a$

(e) $0a = o$

(f) $\alpha 0 = 0$

Theorem **Section Formula**



Let A,B and R be three collinear points.

If $\dfrac{AR}{RB} = \dfrac{m}{n}$, then $\overrightarrow{OR} = \dfrac{m\overrightarrow{OB} + n\overrightarrow{OA}}{m+n}$.

Example Prove that the diagonals of a parallelogram bisect each other.

Solution

Properties

(a) If a, b are two non-zero vectors, then $a // b$ if and only if $a = mb$ for some $m \in R$.

(b) $|a+b| \le |a| + |b|$, and $|a| - |b| \le |a-b|$

Vectors in Three Dimensions

(a) We define i, j, k are vectors joining the origin O to the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ respectively.

(b) i, j and k are unit vectors. i.e. $|i| = |j| = |k| = 1$.

(c) To each point $P(a,b,c)$ in R^3, there corresponds uniquely a vector $\overrightarrow{OP} = p = ai + bj + ck$ where is called the position vector of P.

(d) $|p| = \sqrt{a^2 + b^2 + c^2}$

(e) $p = \dfrac{ai + bj + ck}{\sqrt{a^2 + b^2 + c^2}}$

(f) Properties : Let $p_1 = x_1 i + y_1 j + z_1 k$ and $p_2 = x_2 i + y_2 j + z_2 k$. Then

(i) $p_1 = p_2$ if and only if $x_1 = x_2, y_1 = y_2$ and $z_1 = z_2$,

(ii) $p_1 + p_2 = (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$

(iii) $\alpha p_1 = \alpha(x_1 i + y_1 j + z_1 k) = \alpha x_1 i + \alpha y_1 j + \alpha z_1 k$

N.B. For convenience, we write $p = (x, y, z)$

Example Given two points $A(6,8,-10)$ and $B(1,-2,0)$.

(a) Find the position vectors of A and . B.

(b) Find the unit vector in the direction of the position vector of A.

(c) If a point P divides the line segment AB in the ration $3:2$, find the coordinates of P.

Example Let $A(0,2,6)$ and $B(4,-2,-8)$

(a) Find the position vectors of A and B. Hence find the length of AB.

(b) If P is a point on AB such that $AP = 2PB,$ find the coordinates of P.

(c) Find the unit vector along OP.

Linear Combination and Linear Independence

Definition Consider a given set of vectors $v_1, v_2, \ldots, v_n$. A sum of the form

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

where $a_1, a_2, \ldots, a_n$ are scalars, is called a ***linear combination*** of $v_1, v_2, \ldots, v_n$.
If a vector v can be expressed as $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$
Then v is a linear combination of $v_1, v_2, \ldots, v_n$..

Example $\boldsymbol{r} = \boldsymbol{u} - 2\boldsymbol{v} + \boldsymbol{w}$ is a linear combination of the vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

Example Consider $\boldsymbol{u} = (1,2,-1), \boldsymbol{v} = (6,4,2) \in \boldsymbol{R}^3$, show that $\boldsymbol{w} = (9,27)$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$ while $\boldsymbol{w}_1 = (4,-1,8)$ is not.

Definition If $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ are vectors in $\boldsymbol{R}^n$ and if every vector in $\boldsymbol{R}^n$ can be expressed as the linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$. Then we say that these ***vectors span (generate)*** $\boldsymbol{R}^n$ or $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is the set of the basis vector.

Example $\{\boldsymbol{i}, \boldsymbol{j}\}$ is the set of basis vectors in $\boldsymbol{R}^2$.

Example $\{(1,0,0), (0,1,0), (0,0,1)\}$ is the set basis vector in $\boldsymbol{R}^3$.

Remark :The basis vectors have an important property of linear independent which is defined as follow:

Definition The set of vector $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is said to be ***linear independent*** if and only if the vectors equation $\boldsymbol{k}_1\boldsymbol{v}_1 + \boldsymbol{k}_2\boldsymbol{v}_2 + \cdots + \boldsymbol{k}_n\boldsymbol{v}_n = 0$ has only solution $\boldsymbol{k}_1 = \boldsymbol{k}_2 = \cdots = \boldsymbol{k}_n = 0$

Definition The set of vector $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is said to be ***linear dependent*** if and only if the vectors equation $\boldsymbol{k}_1\boldsymbol{v}_1 + \boldsymbol{k}_2\boldsymbol{v}_2 + \cdots + \boldsymbol{k}_n\boldsymbol{v}_n = 0$ has non-trivial solution.
(i.e. there exists some $\boldsymbol{k}_i$ such that $\boldsymbol{k}_i \neq 0$)

Example Determine whether $\boldsymbol{v}_1 = (1,-2,3), \boldsymbol{v}_2 = (5,6,-1), \boldsymbol{v}_3 = (3,2,1)$ are linear independent or dependent.

Example Let $a = i + j + k, b = 2i - j - k$ and $c = j - k$. Prove that

(a) a, b and c are linearly independent.

(b) any vector v in R^3 can be expressed as a linear combination of a, b and c.

Example If vectors a, b and c are linearly independent, show that $a+b$, $b+c$ and $c+a$ are also linearly independent.

Example Let $a = (2,3-t,1), b = (1-t,2,3)$ and $c = (0,4,2-t)$.

(a) Show that b and c are linearly independent for all real values of t.

(b) Show that there is only one real number t so that a, b and c are linearly dependent.
For this value of t, express a as a linear combination of b and c.

Theorem

(1) A set of vectors including the zero vector must be linearly dependent.

(2) If the vector v can be expressed as a linear combination of $v_1, v_2, \ldots v_n$, then the set of vectors $v_1, v_2, \ldots v_n$ and v are linearly dependent.

(3) If the vectors $v_1, v_2, \ldots v_n$ are linearly dependent, then one of the vectors can expressed as a linear combination of the other vectors.

Example Let $a = i + 3j + 5k,\ \ b = i$ and $c = 3j + 5k$.
Prove that a, b and c are linearly dependent.

Theorem Two non-zero vectors are linearly dependent if and only if they are parallel.

Theorem Three non-zero vectors are linearly dependent if and only if they are coplanar.

Products of Two Vectors

A. Scalar Product

Definition The ***scalar product*** or ***dot product*** or ***inner product*** of two vectors a and b, denoted by $a \cdot b$, is defined as $a \cdot b = |a||b|\cos\theta \quad (0 \le \theta \le \pi)$

where θ is the angle between a and b.

Remarks By definition of dot product, we can find θ by $\cos\theta = \dfrac{a \cdot b}{|a||b|}$.

Example If $|a| = 3, |b| = 4$ and angle between a and b is $60°$, then

$$a \cdot b = 6$$

Theorem Properties of Scalar Product

Let a, b, c be three vectors and m be a scalar. Then we have

(1) $a \cdot a = |a|^2$

(2) $a \cdot b = b \cdot a$

(3) $a \cdot (b + c) = a \cdot b + a \cdot c$

(4) $m(a \cdot b) = (ma) \cdot b = a \cdot (mb)$

(5) $a \cdot a > 0$ if $a \neq 0$ and $a \cdot a = 0$ if $a = 0$

Theorem If $p = a_1 i + b_1 j + c_1 k$ and $q = a_2 i + b_2 j + c_2 k$. Then

(1) $p \cdot q = a_1 a_2 + b_1 b_2 + c_1 c_2$

(2) $\cos\theta = \dfrac{p \cdot q}{|p||q|} \quad (p, q \neq 0)$

$$= \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(3) $p \cdot q = 0$ if and only if $p \perp q$.

(4) $a_1a_2 + b_1b_2 + c_1c_2 = 0$ if and only if $p \perp q$.

Example Find the angle between the two vectors $a = 2i + 2j - k$ and $b = 2i - 2k$.

Remarks Two non-zero vectors are said to be ***orthogonal*** if their scalar product is zero.

Obviously, two perpendicular vectors must be orthogonal since $\theta = \frac{\pi}{2}$, $\cos\theta = 0$, and so their scalar product is zero. For example, as i, j and k are mutually perpendicular, we have

$$i \cdot j = j \cdot k = k \cdot i = 0.$$

Also, as i, j and k are unit vectors, $i \cdot i = j \cdot j = k \cdot k = 1$.

Example State whether the two vectors $i - 3j + 4k$ and $-i + j + k$ are orthogonal.

Example Given two points $A = (2s, -s+1, s+3)$ and $B = (t-2, 3t-1, t)$ and two vectors

$$r_1 = 2i + 2j - k \text{ and } r_2 = -i + j + 2k$$

If $\overrightarrow{AB}$ is perpendicular to both r_1 and r_2, find the values of s and t.

Example Let a, b and c be three coplanar vectors. If a and b are orthogonal, show that

$$c = \frac{c \cdot a}{a \cdot a}a + \frac{c \cdot b}{b \cdot b}b$$

Example Determine whether the following sets of vectors are orthogonal or not.

(a) $a = 4i - 2j$ and $b = 2i + 3j$

(b) $a = 5i - 2j + 4k$ and $b = i + 2j - k$

(c) $a = 3i + j - 4k$ and $b = 2i + 2j + 2k$

Vector Product

Definition If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are vectors in R^3, then the ***vector product*** and ***cross product*** $u \times v$ is the vector defined by

$$u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

$$= \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example Find $a \times b$, $a \cdot (a \times b)$ and $b \cdot (a \times b)$ if $a = 3i + 2j - k$ and $b = i + 4j + k$.

Example Let $a=-i+k, b=2i+j-k$ and $c=i+2j-2k.$ Find

(a) $a \cdot b$ (b) $b \cdot c$ (c) $a \times b$

(d) $a \times c$

(e) $(a \cdot b)c$ (f) $a \cdot (b \times c)$

(g) $a \times b + b \times c + c \times a$ (h) $(a \times b) \cdot c - (c \times b) \cdot a$

(i) $[(a+b) \times c] \cdot a$ (j) $[(a+b) \times (c+a)] \cdot b$

(k) $a \times (b \times c)$ (l) $(a \times b) \times c$

Theorem If u and v are vectors, then

(a) $u \cdot (u \times v) = 0$

(b) $v \cdot (u \times v) = 0$

(c) $|u \times v|^2 = |u|^2 |v|^2 - (u \cdot v)^2$

Proof

Remarks (i) By (c) $|u \times v|^2 = |u|^2 |v|^2 - (u \cdot v)^2$

$= |u|^2 |v|^2 - |u|^2 |v|^2 \cos^2 \theta$, where θ is angle between u and v.

$= |u|^2 |v|^2 (1 - \cos^2 \theta) = |u|^2 |v|^2 \sin^2 \theta$

$\therefore |u \times v| = |u||v| \sin \theta$

The another definition of $u \times v$ is $\boxed{u \times v = |u||v| \sin \theta \, e_n}$ where e_n is a unit vector perpendicular to the plane containing u and v.

(ii) $u \times v = -v \times u$ and $|u \times v| = |v \times u|$

(iii) $i \times j =$ $j \times k =$ $k \times j =$

Definition The ***vector product (cross product)*** of two vectors a and b, denoted by $a \times b$, is a vector such that (1) its magnitude is equal to $|a||b| \sin \theta$, where θ is angle between a and b.

(2) perpendicular to both a and b and $a, b, a \times b$ form a right-hand system.

If a unit vector in the direction of $a \times b$ is denoted by e_n, then we have

$$a \times b = |a||b| \sin \theta \, e_n \qquad (0 \le \theta \le \pi)$$

Geometrical Interpretation of Vector Product

(1) $a \times b$ is a vector perpendicular to the plane containing a and b.

(2) The magnitude of the vector product of a and b is equal to the area of parallelogram with a and b as its adjacent sides.

Corollary (a) Two non-zero vectors are parallel if and only if their vector product is zero.

(b) Two non-zero vectors are linearly dependent if and only if their vector product is zero.

Theorem Properties of Vector Product

(1) $a \times (b + c) = a \times b + a \times c$

(2) $m(a \times b) = (ma) \times b = a \times (mb)$

Example Find a vector perpendicular to the plane containing the points $A(1,2,3)$, $B(-1,4,8)$ and $C(5,1,-2)$.

Example If $a + b + c = 0$, show that $a \times b = b \times c = c \times a$

Example Find the area of the triangle formed by taking $A(0,-2,1), B(1,-1,-2)$ and $C(-1,1,0)$ as vertices.

Example Let $\overrightarrow{OA} = i + 2j + k$, $\overrightarrow{OB} = 3i + j + 2k$ and $\overrightarrow{OC} = 5i + j + 3k$.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(b) Find the area of ΔABC.

Hence, or otherwise, find the distance from C to AB.

Example Let a and b be two vectors in R^3 such that $a \cdot a = b \cdot b = 1$ and $a \cdot b = 0$

Let $S = \{\alpha a + \beta b \in R^3 : \alpha, \beta \in R\}$.

(a) Show that for all $u \in S$, $u = (u \cdot a)a + (u \cdot b)b$

(b) For any $v \in R^3$, let $w = (v \cdot a)a + (v \cdot b)b$. Show that for all $u \in S, (v - w) \cdot u = 0$.

Example Let $a, b, c \in R^3$.

If $a \times (b \times c) = (a \times b) \times c = 0$, prove that $a \cdot b = b \cdot c = c \cdot a = 0$.

Example Let u, v and w be linearly independent vectors in R^3. Show that :

(a) If $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$, then $\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \neq 0$

(b) If $s \in R^3$ such that $s \cdot u = s \cdot v = s \cdot w = 0$, then $s = 0$.

(c) If $u \times (v \times w) = (u \times v) \times w = 0$, then $u \cdot v = v \cdot w = w \cdot u = 0$.

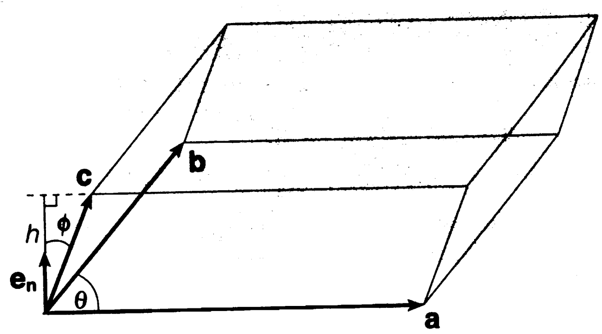
(d) If $u \cdot v = v \cdot w = w \cdot u = 0$,

then $r = \dfrac{r \cdot u}{u \cdot u} u + \dfrac{r \cdot v}{v \cdot v} v + \dfrac{r \cdot w}{w \cdot w} w$ for all $r \in R^3$.

Scalar Triple Product

Definition The ***scalar triple product*** of 3 vectors a, b and c is defined to be $(a \times b) \cdot c$.

Let the angle between a and b be θ and that between $a \times b$ and c be ϕ. As shown in Figure, when $0 < \phi < \frac{\pi}{2}$, we have



Volume of Parallelepiped = $Base\ Area \times Height$

Geometrical Interpretation of Scalar Triple Product

The absolute value of the scalar triple product $(a \times b) \cdot c$ is equal to the volume of the parallelepiped with

a, b and c as its adjacent sides.

Let a, b and c be three vectors. Then

$$(a \times b) \cdot c = (b \times c) \cdot a = (c \times a) \cdot b$$

Remarks Volume of Parallelepiped = $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Example Let $A(3,-5,6)$, $B(2,3,-2)$, $C(-1,8,-8)$

(a) Find the volume of parallelepiped with sides $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OC}$.

(b) What is the geometrical relationship about point O, A, B, C in (a).

Example A, B, C are the points $(1,1,0)$, $(2,-1,1)$, $(-1,-1,1)$ respectively and O is the origin.

Let $a = \overrightarrow{OA}, b = \overrightarrow{OB}$ and $c = \overrightarrow{OC}$.

(a) Show that a, b and c are linearly independent.

(b) Find

(i) the area of $\triangle OAB$, and

(ii) the volume of tetrahedron $OABC$.

Solution

Matrix Transformation*

<u>Linear transformation of a plane (reflections, rotation)</u>

Consider the case with the point $P(x, y) \to P'(x', y')$ such that $x = x', y = y'$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$r' = Ar, \qquad \text{where } A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

A is a matrix of transformation of reflection.

In general, any column vector pre-multiplied by a 2×2 matrix, it is transformed or mapped (x', y') into another column vector.

Example $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

We have $x' = ax + by$

$y' = cx + dy$

If using the base vector in R^2, i.e $(1,0),(0,1)$.

$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$

then a, b, c, d can be found.

The images of the points $(1,0),(0,1)$ under a certain transformation are known.

Therefore, the matrix is known.

Eight Simple Transformation

I. Reflection in x-axis

II. Reflection in y-axis

III. Reflection in $x = y$.

IV. Reflection in the line $y = -x$

V. Quarter turn about the origin

VI. Half turn about the origin

VII. Three quarter turn about the origin

VIII. Identity Transformation

Some Special Linear Transformations on R^2

I. Enlargement

If $\left|\overrightarrow{OP}\right| = r$, then $\left|\overrightarrow{OP'}\right| = kr$.

$$A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

II. (a) Shearing Parallel to the x-axis

The y-coordinate of a point is unchanged but the x-coordinate is changed by adding to it

(a) quantity which is equal to a multiple of the value of its y-coordinate.

(b) Shearing Parallel to the y-axis

III. Rotation

IV. Reflection about the line $y = (\tan\alpha)x$

Example If the point $P(4,2)$ is rotated clockwise about the origin through an angle $60°$, find its final position

Solution

Example A translation on R^2 which transforms every point P whose position vector is $p = \begin{pmatrix} x \\ y \end{pmatrix}$

To another point Q with position vector $q = \begin{pmatrix} x' \\ y' \end{pmatrix}$ defined by $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Find the image of (a) the point $(-4,2)$ (b) the line $2x + y = 0$

Linear Transformation

Definition Let V and U be two sets. A mapping $\sigma : V \to U$ is called a ***linear transformation*** from V to U if and only if it satisfies the condition:

$$\sigma(au + bv) = a\sigma(u) + b\sigma(v), \forall u, v \in V \text{ and } \forall a, b \in R.$$

Example Let V be the set of 3×1 matrices and A be any real 3×3 matrix. A mapping $f : V \to V$ Such that $f(x) = Ax, \forall x \in V$. Show that f is linear.

In R^3, consider a linear transformation $\sigma : R^3 \to R^3$, let $v \in R^3$, $v = (a,b,c) = ai + bj + ck$.

We are going to find the image of v under σ.

$$\sigma(v) = \sigma(ai + bj + ck) = a\sigma(i) + b\sigma(j) + c\sigma(k)$$

Therefore, $\sigma(v)$ can be found if $\sigma(i), \sigma(j)$ and $\sigma(k)$ are known. That is to say, to specify σ completely, it is only necessary to define $\sigma(i), \sigma(j)$ and $\sigma(k)$.

For instance, we define a linear transformation

$\sigma : R^3 \to R^3$ by $\sigma(i) = 2i - j - 3k, \sigma(j) = i + 2k, \sigma(k) = 3i - 2j + 2k$.

$\therefore \quad \sigma(3i + 2j - 4k) =$

$= -4i + 5j - 13k$

We form a matrix A such that $A = \begin{pmatrix} \sigma(i) & \sigma(j) & \sigma(k) \end{pmatrix}$

$$= \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & -2 \\ -3 & 2 & 2 \end{pmatrix}$$

Consider $A\begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & -2 \\ -3 & 2 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ -13 \end{pmatrix}$

The result obtained is just the same as $\sigma(3i + 2j - 4k)$.

The matrix A representing the linear transformation σ is called the ***matrix representation of the linear transformation*** σ

Example Let $\sigma : R^3 \to R^2$, defined by $\sigma(i) = i + 2j, \sigma(j) = -j, \sigma(k) = 4i - 3j.$

The matrix represent representation of a linear transformation is $\begin{pmatrix} 1 & 0 & 4 \\ 2 & -1 & -3 \end{pmatrix}_{2\times 3}$.

Example The matrix $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}$ represents a linear transformation

$\sigma : R^2 \to R^3$, defined by $\sigma(i) = i + k, \sigma(j) = 2i - j + k$.

Example Let $\sigma, \tau : R^3 \to R^3$ be two linear transformations whose matrix representations are respectively

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix}.$$

Find the matrix representation of $\sigma \circ \tau$.

Example If $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ for any $(x, y) \in R^2$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is said to be the matrix representation of the transformation which transforms (x, y) to (x', y').

Find the matrix representation of

(a) the transformation which transforms any point (x, y) to $(-x, y)$,

(b) the transformation which transforms any point (x, y) to (y, x)

Example It is given that the matrix representing the reflection in the line $y = (\tan\alpha)x$ is

$$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

Let T be the reflection in the line $y = \frac{1}{2}x$.

(a) Find the matrix representation of T.

(b) The point $(4,7)$ is transformed by T to another point (x_1, y_1). Find x_1, y_1.

(c) The point $(4,10)$ is reflected in the line $y = \frac{1}{2}x + 3$ to another point (x_2, y_2).

Find x_2 and y_2.