BCA 2nd Year

Optimization Techniques

UNIT – 1

Linear Programming

Ques – A tyre factory produces 3 types of tyres T_1, T_2, T_3. Three different types of chemicals say C_1, C_2 and C_3 are required for production. One T_1 needs 2 units of C_1, 3 units of C_3. One T_2 tyre needs 3 units of C_1, 2 units of C_2 and 2 units of C_3 and one T_3 tyre needs 5 units of C_2 and 4 units of C_3. The factory has only one stock of 20 unit of C_1, 25 units of C_2 and 30 units of C_3. Further the profit from the sale of one tyre T_1 is 6 Rs., one tyre of T_2 is 10 Rs., one tyre T_3 is 8 Rs. Assuming that the factory can sell all that it produces formulate a L.P.P. to maximize its profit.

Sol:

	Tyre type T_1	Tyre type T_2	Tyre type T_3	Total units in store
chemical C_1	2	3	0	20
chemical C_2	0	2	5	25
chemical C_3	3	2	4	30
Profit	6 Rs.	10 Rs.	8 Rs.	

Type $T_1 \Rightarrow x_1$
Type $T_2 \Rightarrow x_2$
Type $T_3 \Rightarrow x_3$

Objective function –

$$\max z = 6x_1 + 10x_2 + 8x_3$$

Subject to –
$$2x_1 + 3x_2 \le 20$$
$$2x_2 + 5x_3 \le 25$$
$$3x_1 + 2x_2 + 4x_3 \le 30$$

where $x_1, x_2, x_3 \ge 0$.

Ques - A seller buy some tables and chairs. He has 5000 Rs. to invest and a space to store at most 60 piece. A table cost him 250 Rs. and a chair 50 Rs. He can sell a table at a profit of 50 Rs. and a chair at a profit of 15 Rs. Assuming that he can sell all the pieces that he buy. Prepare a mathematical formulation of this linear programming problem to determine the number of pieces of each type to gain maximum profit.

	Table	Chair	Total units
Cost	250	50	5000
Profit	50	15	

Table $\Rightarrow x_1$

Chair $\Rightarrow x_2$

Objective function -

$$\max z = 50x_1 + 15x_2$$

Subject to -

$$250x_1 + 50x_2 \leq 5000$$

M. Imp

★ Simplex method :-

If $\leq \Rightarrow +S$ (Slack variable)

If $\geq \Rightarrow -S$ (Surplus variable)

Ques- max $z = 4x_1 + 5x_2$

Subject to : $2x_1 + 3x_2 \le 24$

$2x_1 + x_2 \le 16$

$x_1, x_2 \ge 0$

Sol: max $z = 4x_1 + 5x_2 + 0S_1 + 0S_2$

Subject to :

$2x_1 + 3x_2 + S_1 + 0S_2 = 24$

$2x_1 + x_2 + 0S_1 + S_2 = 16$

Key element

	C_j		4	5	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	X_B/X_j
outgoing vector ← S_1	0	24	2	$\boxed{3}$	1	0	$24/3 = 8$ → min ratio
S_2	0	16	2	1	0	1	$16/1 = 16$
Δ_j	$z=0$		-4	-5	0	0	

→ most negative

$\Delta_j = C_B X_j - C_j$

$\Delta_1 = (0,0)(2,2) - 4$
$= -4$

$\Delta_2 = (0,0)(3,1) - 5$
$= -5$

$\Delta_3 = (0,0)(1,0) - 0$
$= 0$

$\Delta_4 = (0,0)(0,1) - 0$
$= 0$

$R_1 \to R_1/3$, $R_2 \to R_2 - R_1$

incoming vector Key element

	C_j		4	5	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	
x_2	5	8	$2/3$	1	$1/3$	0	$8 \times \frac{3}{2} = 12$
outgoing vector ← S_2	0	8	$\boxed{4/3}$	0	$-1/3$	1	$8 \times \frac{3}{4} = 6$ → min ratio
Δ_j	$z = 40$		$-2/3$	0	$5/3$	0	

→ most negative

$\Delta_1 = (5,0)(2/3, 4/3) - 4$
$= 10/3 - 4 = -2/3$

$\Delta_2 = (5,0)(1,0) - 5$
$= 5-5 = 0$

$\Delta_3 = (5,0)(1/3, -1/3) - 0$
$= 5/3$

$\Delta_4 = (5,0)(0,1) - 0$
$= 0$

$R_1 \rightarrow R_1 - 2/3 R_2$

	C_j		4	5	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2
x_2	5	4	0	1	1/2	-1/2
x_1	4	6	1	0	-1/4	3/4
Δ_j	$z = 44$		0	0	3/2	1/2

$\Delta_1 = (5,4)(0,1) - 4$
$= 4-4 = 0$

$\Delta_2 = (5,4)(1,0) - 5$
$= 5-5 = 0$

$\Delta_3 = (5,4)(1/2, -1/4) - 0$
$= 5/2 - 1 = 3/2$

$\Delta_4 = (5,4)(-1/2, 3/4) - 0$
$= -5/2 + 3 = 1/2$

$\Delta_j \geq 0$, the solution is optimum.
max $z = 44$
$x_1 = 6$
$x_2 = 4$

Ques – Solve by simplex method –
Max $Z = 4x_1 + 6x_2 + 2x_3$
subject to

$x_1 + x_2 + x_3 \leq 3$
$x_1 + 4x_2 + 7x_3 \leq 9$
$x_1, x_2, x_3 \geq 0$

Sol: $\text{Max } Z = 4x_1 + 6x_2 + 2x_3 + 0S_1 + 0S_2$

subject to: $x_1 + x_2 + x_3 + S_1 + 0S_2 = 3$

$x_1 + 4x_2 + 7x_3 + 0S_1 + S_2 = 9$

$x_1, x_2, x_3 \geq 0$

incoming vector

	C_j		4	6	2	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	X_B/X_j
S_1	0	3	1	1	1	1	0	$\frac{3}{1} = 3$
outgoing vector ← S_2	0	9	1	$\boxed{4}$	7	0	1	$\frac{9}{4} = 2.2$ → min ratio
Δ_j	$Z = 0$		-4	-6	-2	0	0	

key element most negative

$\Delta_1 = (0,0)(1,1) - 4$
$= -4$

$\Delta_2 = (0,0)(1,4) - 6$
$= -6$

$\Delta_3 = (0,0)(1,7) - 2$
$= -2$

$\Delta_4 = (0,0)(1,0) - 0 = 0$

$\Delta_5 = (0,0)(0,1) - 0 = 0$

$R_2 \to R_2/4, \quad R_1 \to R_1 - R_2$

incoming vector

	C_j		4	6	2	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	X_B/X_j
outgoing vector → S_1	0	3/4	$\boxed{3/4}$	0	$-3/4$	1	$-1/4$	$\frac{3}{4} \div \frac{3}{4} = 1$ → min ratio
x_2	6	9/4	1/4	1	7/4	0	1/4	$\frac{9}{4} \div \frac{1}{4} = 9$
Δ_j	$Z = 27/2$		$-5/2$	0	17/2	0	3/2	

most neg

$\Delta_1 = (0,6)(3/4, 1/4) - 4$
$= 6/4 - 4 = -5/2$

$\Delta_2 = (0,6)(0,1) - 6$
$= 6 - 6 = 0$

$\Delta_3 = (0,6)(-3/4, 7/4) - 2$
$= 42/4 - 2 = 17/2$

$\Delta_4 = (0,6)(1,0) - 0$
$= 0$

$\Delta_5 = (0,6)(-1/4, 1/4) - 0$
$= \frac{6}{4} = 3/2$

$R_1 \to R_1 \times 4/3$, $R_2 \to R_2 - \frac{1}{4} R_1$

	C_j		4	6	2	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2
x_1	4	1	1	0	-1	3/4	-3/16
x_2	6	2	0	1	2	-1/4	0
Δ_j	Z = 16		0	0	6	3/2	-3/4

$\Delta_1 = (4,6)(1,0) - 4$
$= 4 - 4 = 0$

$\Delta_2 = (4,6)(0,1) - 6$
$= 6 - 6 = 0$

$\Delta_3 = (4,6)(-1,2) - 2$
$= -4 + 12 - 2 = 6$

$\Delta_4 = (4,6)(3/4, -1/4) - 0$
$= 3 - 6/4 = 3/2$

$\Delta_5 = (4,6)(-3/16, 0) - 0$
$= -3/4$

$$\boxed{\begin{array}{l} \text{Max } Z = 16 \\ x_1 = 1,\ x_2 = 2,\ x_3 = 3 \end{array}}$$

Ques- Solve the following L.P.P. by simplex method-

$$\text{Max } Z = 3x_1 + 2x_2$$

subject to:
$$x_1 + x_2 \le 4$$
$$x_1 - x_2 \le 2$$
$$x_1, x_2 \ge 0$$

Sol: $\text{Max } z = 3x_1 + 2x_2 + 0S_1 + 0S_2$

subject to:
$$x_1 + x_2 + S_1 + 0S_2 = 4$$
$$x_1 - x_2 + 0S_1 + S_2 = 2$$
$$x_1, x_2, S_1, S_2 \ge 0$$

Incoming vector

B.V.	C_B	X_B	x_1	x_2	S_1	S_2	Ratio X_B/X_j	
	C_j		3	2	0	0		
S_1	0	4	1	1	1	0	$\frac{4}{1}=4$	
outgoing vector $\rightarrow S_2$	0	2	$\boxed{1}$	−1	0	1	$\frac{2}{1}=2$	$\rightarrow$ min ratio
Δ_j	$Z=0$		−3	−2	0	0		

most neg   Key element

$\Delta_1 = (0,0)(1,1)-3 = -3$

$\Delta_2 = (0,0)(1,-1)-2 = -2$

$\Delta_3 = (0,0)(1,0)-0 = 0$

$\Delta_4 = (0,0)(0,1)-0 = 0$

$R_1 \rightarrow R_1 - R_2$

incoming vector   key element

B.V.	C_B	X_B	x_1	x_2	S_1	S_2	Ratio X_B/X_j	
	C_j		3	2	0	0		
outgoing vector $\rightarrow S_1$	0	2	0	$\boxed{2}$	1	−1	$\frac{2}{2}=1$	$\rightarrow$ min ratio
x_1	3	2	1	−1	0	1	$\frac{2}{-1}=-2$ (neglect)	
Δ_j	$Z=6$		0	−5	0	3		

most negative

$\Delta_1 = (0,3)(0,1)-3 = 3-3 = 0$

$\Delta_2 = (0,3)(2,-1)-2 = -3-2 = -5$

$\Delta_3 = (0,3)(1,0)-0 = 0$

$\Delta_4 = (0,3)(-1,1)-0 = 3$

$R_1 \rightarrow R_1/2\,,\quad R_2 \rightarrow R_2 + R_1$

B.V.	C_B	X_B	x_1	x_2	S_1	S_2
	C_j		3	2	0	0
x_2	2	1	0	1	1/2	−1/2
x_1	3	3	1	0	1/2	1/2
Δ_j	$Z=11$		0	0	5/2	1/2

$\Delta_1 = (2,3)(0,1)-3 = 3-3 = 0$

$\Delta_2 = (2,3)(1,0)-2 = 2-2 = 0$

$\Delta_3 = (2,3)(1/2, 1/2) - 0$
$= 1 + 3/2 = 5/2$

$\Delta_4 = (2,3)(-1/2, 1/2) - 0$
$= -1 + 3/2 = 1/2$

$$\boxed{\begin{array}{l} Z = 11 \\ x_1 = 3, \quad x_2 = 1 \end{array}}$$

Ques – Max $z = 40x_1 + 60x_2$

subject to:
$3x_1 + 3x_2 \le 36$
$5x_1 + 2x_2 \le 60$
$2x_1 + 6x_2 \le 60$

Sol: Max $z = 40x_1 + 60x_2 + 0S_1 + 0S_2 + 0S_3$

subject to:

$3x_1 + 3x_2 + S_1 + 0S_2 + 0S_3 = 36$
$5x_1 + 2x_2 + 0S_1 + S_2 + 0S_3 = 60$
$2x_1 + 6x_2 + 0S_1 + 0S_2 + S_3 = 60$

incoming vector

		C_j	40	60	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3	X_B/x_j
S_1	0	36	3	3	1	0	0	$\frac{36}{3} = 12$
S_2	0	60	5	2	0	1	0	$\frac{60}{2} = 30$
Outgoing vector ← S_3	0	60	2	$\boxed{6}$	0	0	1	$\frac{60}{6} = 10$ → min ratio
Δ_j	$Z = 0$		-40	-60	0	0	0	

Key element

most negative

$\Delta_1 = (0,0,0)(3,5,2) - 40$
$= -40$

$\Delta_2 = (0,0,0)(3,2,6) - 60$
$= -60$

$\Delta_3 = (0,0,0)(1,0,0) - 0$
$= 0$

$\Delta_4 = (0,0,0)(0,1,0) - 0 = 0$

$\Delta_5 = (0,0,0)(0,0,1) - 0 = 0$

$R_1 \to R_1 - 3R_3$, $R_3 \to R_3/6$, $R_2 \to R_2 - 2R_3$

incoming vector   Key element

B.V.	C_B	X_B	40 x_1	60 x_2	0 S_1	0 S_2	0 S_3	Ratio X_B/x_j
S_1	0	4	$\boxed{2}$	0	1	0	-1/2	$\frac{6}{2} = 3$ → min. ratio
S_2	0	40	13/3	0	0	1	-1/3	$40 \times \frac{3}{13} = 9.2$
x_2	60	10	1/3	1	0	0	1/6	$\frac{10}{3} = 3.3$
Δj		Z = 600	-20	0	0	0	10	

Outgoing vector → S_1

most negative

$\Delta_1 = (0,0,60)(2,13/3,1/3) - 40$
$= 20 - 40 = -20$

$\Delta_2 = (0,0,60)(0,0,1) - 60$
$= 60 - 60 = 0$

$\Delta_3 = (0,0,60)(1,0,0) - 0$
$= 0$

$\Delta_4 = (0,0,60)(0,1,0) - 0$
$= 0$

$\Delta_5 = (0,0,60)(-1/2,-1/3,1/6) - 0$
$= 10$

$R_1 \to R_1/2$, $R_2 \to R_2 - \frac{13}{3}R_1$, $R_3 \to R_3 - \frac{1}{3}R_1$

B.V.	C_B	X_B	40 x_1	60 x_2	0 S_1	0 S_2	0 S_3
x_1	40	3	1	0	1/2	0	-1/4
S_2	0	27	0	0	-13/6	1	3/4
x_2	60	9	0	1	-1/6	0	1/4
Δj		Z = 660	0	0	10	0	5

$\Delta_1 = 40 - 40 = 0$

$\Delta_2 = 60 - 60 = 0$

$\Delta_3 = 20 - 10 = 10$

$\Delta_4 = 0$

$\Delta_5 = -10 + 15 = 5$

$$\boxed{\begin{array}{l} \max Z = 660 \\ x_1 = 3,\ x_2 = 9,\ x_3 = 0 \end{array}}$$

Imp

Ques – Solve the following L.P.P. by simplex method –

$$\max z = 3x_1 + 5x_2 + 4x_3$$

Subject to:

$$2x_1 + 3x_2 \le 8$$
$$0x_1 + 2x_2 + 5x_3 \le 10$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$
$$x_1, x_2, x_3 \ge 0$$

Sol:

$$\max z = 3x_1 + 5x_2 + 4x_3 + 0S_1 + 0S_2 + 0S_3$$

Subject to:

$$2x_1 + 3x_2 + 0x_3 + S_1 + 0S_2 + 0S_3 = 8$$
$$0x_1 + 2x_2 + 5x_3 + 0S_1 + S_2 + 0S_3 = 10$$
$$3x_1 + 2x_2 + 4x_3 + 0S_1 + 0S_2 + S_3 = 15$$

Incoming vector   Key element

	C_j		3	5	4	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/x_j
Outgoing vector → S_1	0	8	2	$\boxed{3}$	0	1	0	0	$\frac{8}{3} = 2.8$ → min ratio
S_2	0	10	0	2	5	0	1	0	$\frac{10}{2} = 5$
S_3	0	15	3	2	4	0	0	1	$\frac{15}{2} = 7.5$
Δ_j	$z = 0$		-3	-5	-4	0	0	0	

most negative

$\Delta_1 = (0,0,0)(2,0,3) - 3$
$= -3$

$\Delta_2 = (0,0,0)(3,2,2) - 5$
$= -5$

$\Delta_3 = (0,0,0)(0,5,4) - 4$
$= -4$

$\Delta_4 = (0,0,0)(1,0,0) - 0$
$= 0$

$\Delta_5 = (0,0,0)(0,1,0) - 0$
$= 0$

$\Delta_6 = (0,0,0)(0,0,1) - 0$
$= 0$

$$R_1 \to \frac{R_1}{3}, \quad R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 2R_1$$

B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	Ratio
	C_j		3	5	4	0	0	0	
x_2	5	8/3	2/3	1	0	1/3	0	0	$\frac{8}{3} \times \frac{1}{0} = \infty$
S_2	0	14/3	-4/3	0	$\boxed{5}$	-2/3	-1	0	$14/15 = 0.9$ → min
S_3	0	29/3	5/3	0	4	-2/3	0	1	$29/4 = 2.4$
Δ_j	$Z = 40/3$		1/3	0	-4	+5/3	0	0	

most neg.

$\Delta_1 = (5,0,0)(2/3, -4/3, 5/3) - 3$
$= 10/3 - 3 = 1/3$

$\Delta_2 = (5,0,0)(1,0,0) - 5$
$= 5 - 5 = 0$

$\Delta_3 = (5,0,0)(0,5,4) - 4$
$= -4$

$\Delta_4 = (5,0,0)(1/3, -2/3, -2/3) - 0$
$= 5/3$

$\Delta_5 = (5,0,0)(0,-1,0) - 0$
$= 0$

$\Delta_6 = (5,0,0)(0,0,1) - 0$
$= 0$

$R_2 \to \frac{R_2}{5}$, $R_3 \to R_3 - 4R_2$

B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	Ratio X_B/X_j
	C_j		3	5	4	0	0	0	
x_2	5	8/3	2/3	1	0	1/3	0	0	$\frac{8}{3} \times \frac{3}{2} = 4$
x_3	4	14/15	-4/15	0	1	-2/15	1/5	0	$\frac{14}{15} \times -\frac{15}{4} = -\frac{14}{4}$
S_3	0	89/15	$\boxed{41/15}$	0	0	-2/15	-4/5	1	$\frac{89}{15} \times \frac{15}{41} = \frac{89}{41}$ → min ratio
Δ_j	$Z = 256/15$		-11/15	0	0	17/15	4/5	0	

most neg.

$\Delta_1 = (5,4,0)(2/3, -4/15, 41/15) - 3$
$= 10/3 - 16/15 - 3 = -11/15$

$\Delta_2 = (5,4,0)(1,0,0) - 5$
$= 5 - 5 = 0$

$\Delta_3 = (5,4,0)(0,1,0) - 4$
$= 4 - 4 = 0$

$\Delta_4 = (5,4,0)(1/3, -2/15, -2/15)$
$= 5/3 - 8/15 = 17/15$

$\Delta_5 = (5,4,0)(0, 1/5, -4/5) = 4/5$

$\Delta_6 = (5,4,0)(0,0,1) = 0$

$R_3 \to R_3 \times \frac{15}{41}$, $R_2 \to R_2 + \frac{4}{15} R_3$, $R_1 \to R_1 - \frac{2}{3} R_3$

	C_j		3	5	4	0	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
x_2	5	50/41	0	1	0	15/41	8/41	-10/41
x_3	4	62/41	0	0	1	-6/41	5/41	4/41
x_1	3	89/41	1	0	0	-2/41	-12/41	15/41
Δ_j	$z = 765/41$		0	0	0	45/41	24/41	11/41

$\Delta_1 = 3-3 = 0$

$\Delta_2 = 5-5 = 0$

$\Delta_3 = 4-4 = 0$

$\Delta_4 = 75/41 - 24/41 - 6/41$
$= 45/41$

$\Delta_5 = 40/41 + 20/41 - 36/41$
$= 24/41$

$\Delta_6 = -50/41 + 16/41 + 45/41$
$= 11/41$

$$\boxed{\begin{array}{l} \max z = \frac{765}{41} \\ x_1 = \frac{89}{41}, \quad x_2 = \frac{50}{41}, \quad x_3 = \frac{62}{41} \end{array}}$$

Ques – Solve the L.P.P. by simplex method –

Minimize $z = x_1 - 3x_2 + 2x_3$

Subject to:

$$3x_1 - x_2 + 3x_3 \leq 7$$
$$-2x_1 + 4x_2 \leq 12$$
$$-4x_1 + 3x_2 + 8x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

Sol: Converting the minimization into maximization

$$\text{mini } z = x_1 - 3x_2 + 2x_3$$
$$\text{Max } (-z) = -x_1 + 3x_2 - 2x_3$$
$$\text{Max } z^* = -x_1 + 3x_2 - 2x_3$$

Subject to:

$$3x_1 - x_2 + 3x_3 + S_1 + 0S_2 + 0S_3 = 7$$

$$-2x_1 + 4x_2 + 0x_3 + 0S_1 + S_2 + 0S_3 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + 0S_1 + 0S_2 + S_3 = 10$$
$$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$$

Key element

	C_j		-1	3	-2	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B / X_j
S_1	0	7	3	-1	3	1	0	0	$\frac{7}{-1} = -7$
Outgoing vector $\rightarrow S_2$	0	12	-2	$\boxed{4}$	0	0	1	0	$\frac{12}{4} = 3 \rightarrow$ min ratio
S_3	0	10	-4	3	8	0	0	1	$\frac{10}{3} = 3.3$
Δ_j	$Z^* = 0$		-1	-3	2	0	0	0	

most negative

$\Delta_1 = (0\ 0\ 0)(3\ -2\ -4) + 1$
$= +1$

$\Delta_2 = (0\ 0\ 0)(-1\ 4\ 3) - 3$
$= -3$

$\Delta_3 = (0\ 0\ 0)(3\ 0\ 8) + 2$
$= 2$

$\Delta_4 = (0\ 0\ 0)(1\ 0\ 0) - 0$
$= 0$

$\Delta_5 = (0\ 0\ 0)(0\ 1\ 0) - 0$
$= 0$

$\Delta_6 = (0\ 0\ 0)(0\ 0\ 1) - 0$
$= 0$

$R_2 \rightarrow \frac{R_2}{4}$, $R_1 \rightarrow R_1 + R_2$, $R_3 \rightarrow R_3 - 3R_2$

	C_j		-1	3	-2	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B / X_j
Outgoing vector $\rightarrow S_1$	0	10	$\boxed{5/2}$	0	3	1	1/4	0	$10 \times \frac{2}{5} = 4 \rightarrow$ min ratio
x_2	3	3	-1/2	1	0	0	1/4	0	$3 \times -\frac{2}{1} = -6$
S_3	0	1	-11/2	0	8	0	-3/4	1	$1 \times -\frac{2}{11} = -2/11$
Δ_j	$Z^* = 9$		-1/2	0	2	0	3/4	0	

most neg.

$\Delta_1 = (0\ 3\ 0)(5/2\ -1/2\ -11/2) + 1$
$= -3/2 + 1 = -1/2$

$\Delta_2 = (0\ 3\ 0)(0\ 1\ 0) - 3$
$= 3 - 3 = 0$

$\Delta_3 = (0\ 3\ 0)(3\ 0\ 8) + 2$
$= 2$

$\Delta_4 = (0\ 3\ 0)(1\ 0\ 0) - 0$
$= 0$

$\Delta_5 = (0\ 3\ 0)(1/4\ 1/4\ -3/4) - 0$
$= 3/4$

$\Delta_6 = (0\ 3\ 0)(0\ 0\ 1) - 0$
$= 0$

$R_1 \to R_1 \times \frac{2}{5}$, $R_2 \to R_2 + \frac{1}{2} R_1$, $R_3 \to R_3 \times R_2$

	C_j		-1	3	-2	0	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
x_1	-1	4	1	0	6/5	2/5	1/10	0
x_2	3	5	0	1	3/5	1/5	3/10	0
S_3	0	5	0	0	24/5	0	-9/40	0
Δ_j	$Z^* = 11$		0	0	13/5	1/5	4/5	0

$\Delta_1 = (-1\ 3\ 0)(1\ 0\ 0) + 1$
$= -1 + 1 = 0$

$\Delta_2 = (-1\ 3\ 0)(0\ 1\ 0) - 3$
$= 3 - 3 = 0$

$\Delta_3 = (-1\ 3\ 0)(6/5\ 3/5\ 24/5) + 2$
$= -6/5 + 9/5 + 2 = 13/5$

$\Delta_4 = (-1\ 3\ 0)(2/5\ 1/5\ 0) - 0$
$= -2/5 + 3/5 = 1/5$

$\Delta_5 = (-1\ 3\ 0)(1/10\ 3/10\ -9/40)$
$= -1/10 + 9/10 = 4/5$

$\Delta_6 = (-1\ 3\ 0)(0\ 0\ 0) - 0$
$= 0$

$$\boxed{\begin{array}{l} \text{Max } z = -11 \\ x_1 = 4,\ x_2 = 5,\ x_3 = 0 \end{array}}$$

★ **Big-M-Method (Method of penalty)** –

Ques – Mini $z = 4x_1 + 3x_2$

Subject to: $2x_1 + x_2 \geq 10$
$-3x_1 + 2x_2 \leq 6$
$x_1 + x_2 \geq 6$
$x_1, x_2 \geq 0$

Sol: Converting the minimization into maximization

Max $(-z) = z^* = -4x_1 - 3x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_2$

Subject to: $2x_1 + x_2 - S_1 + A_1 = 10$
$-3x_1 + 2x_2 + S_2 = 6$
$x_1 + x_2 - S_3 + A_2 = 6$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

where S_2 is slack variable and S_1, S_3 are surplus variable. A_1 and A_2 are artificial variables.

	C_j		-4	-3	0	0	0	-M	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	X_B/x_j
A_1	-M	10	$\boxed{2}$	1	-1	0	0	1	0	$\frac{10}{2} = 5 \rightarrow$ min
S_2	0	6	-3	2	0	1	0	0	0	$\frac{6}{-3} = -2$
A_2	-M	6	1	1	0	0	-1	0	1	$\frac{6}{1} = 6$
Δ_j	$Z^* = -16M$		$-3M+4$	$-2M+3$	M	0	M	0	0	

most negative

$\Delta_1 = -3M+4$

$\Delta_2 = -2M+3$

$\Delta_3 = M$

$\Delta_4 = 0$

$\Delta_5 = M$

$\Delta_6 = 0$

$\Delta_7 = 0$

$R_1 \rightarrow \frac{R_1}{2}, \quad R_2 \rightarrow R_2 + 3R_1, \quad R_3 \rightarrow R_3 - R_1$

	C_j		-4	-3	0	0	0	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_2	X_B/x_j
x_1	-4	5	1	1/2	-1/2	0	0	0	$5 \times \frac{2}{1} = 10$
S_2	0	21	0	7/2	-3/2	1	0	0	$21 \times \frac{2}{7} = 6$
A_2	-M	1	0	$\boxed{1/2}$	1/2	0	-1	1	$1 \times \frac{2}{1} = 2 \rightarrow$ min ratio
Δ_j	$Z^* = -20-M$		0	$-\frac{M}{2}+1$	$-\frac{M}{2}+2$	0	M	0	

most neg.

$\Delta_1 = 0$

$$\Delta_2 = -2 - \frac{M}{2} + 3 = \frac{-4-M+6}{2} = -\frac{M}{2} + 1$$

$$\Delta_3 = 2 - \frac{M}{2} = -\frac{M}{2} + 2$$

$\Delta_4 = 0$

$\Delta_5 = M$

$\Delta_6 = 0$

$R_3 \to R_3 \times 2$, $R_2 \to R_2 - \frac{7}{2}R_3$, $R_1 \to R_1 - \frac{1}{2}R_3$

	C_j		-4	-3	0	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3
x_1	-4	4	1	0	-1	0	1
S_2	0	14	0	0	-5	1	7
x_2	-3	2	0	1	1	0	-2
Δ_j	$z^* = -22$		0	0	1	0	2

$\Delta_1 = -4+4 = 0$
$\Delta_2 = -3+3 = 0$
$\Delta_3 = 4-3 = 1$
$\Delta_4 = 0$
$\Delta_5 = -4+6 = 2$

$\Delta_j \geq 0$, the solution is optimum.

$$\boxed{\begin{array}{l} \max z^* = -22, \ \text{mini } z = 22 \\ x_1 = 4, x_2 = 2, x_3 = 0, S_2 = 14 \end{array}}$$

Ques- Solve the L.P.P. by big-M-method -

mini $z = 3x_1 + 8x_2$

Subject to:

$$x_1 + x_2 = 200$$
$$x_1 \leq 80$$
$$x_2 \geq 60$$
$$x_1, x_2 \geq 0$$

Sol: Converting the minimization into maximization

$\max z^* = (-z) = -3x_1 - 8x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to: $x_1 + x_2 + A_1 = 200$

$$x_1 + S_1 = 80$$
$$x_2 - S_2 + A_2 = 60$$
$$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$$

$S_1 \to$ Slack variable , $S_2 \to$ Surplus variable

$A_1, A_2 \to$ Artificial variables

incoming vector

	C_j		-3	-8	0	0	-M	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	A_1	A_2	X_B/x_j
A_1	-M	200	1	1	0	0	1	0	200
S_1	0	80	1	0	1	0	0	0	∞ (neglect)
Outgoing vector → A_2	-M	60	0	$\boxed{1}$	0	-1	0	1	60 → min ratio
Δ_j	$Z^* = -260M$		-M+3	-2M+8	0	M	0	0	

most negative

$\Delta_1 = -M+3$

$\Delta_2 = -2M+8$

$\Delta_3 = 0$

$\Delta_4 = M$

$\Delta_5 = 0$

$\Delta_6 = 0$

$R_1 \rightarrow R_1 - R_3$

	C_j		-3	-8	0	0	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	A_1	X_B/x_j
A_1	-M	140	1	0	0	1	1	140
S_1	0	80	$\boxed{1}$	0	1	0	0	80 → min ratio
x_2	-8	60	0	1	0	-1	0	∞ (neglect)
Δ_j	$Z^* = -140M-480$		-M+3	0	0	-M+8	0	

$\Delta_1 = -M+3$

$\Delta_2 = -8+8=0$

$\Delta_3 = 0$

$\Delta_4 = -M+8$

$\Delta_5 = 0$

$R_1 \rightarrow R_1 - R_2$

	C_j		-3	-8	0	0	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	A_1	X_B/x_j
A_1	-M	60	0	0	-1	$\boxed{1}$	1	60 → min
x_1	-3	80	1	0	1	0	0	∞ (neglect)
x_2	-8	60	0	1	0	-1	0	-60 (neglect)
Δ_j	$Z^* = -60M-720$		0	0	M-3	-M+8	0	

$\Delta_1 = -3+3=0$

$\Delta_2 = 0$

$\Delta_3 = M-3$

$\Delta_4 = -M+8$

$\Delta_5 = 0$

$R_3 \to R_3 + R_1$

	C_j		-3	-8	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2
S_2	0	60	0	0	-1	1
x_1	-3	80	1	0	1	0
x_2	-8	120	0	1	-1	0
Δ_j	$z^* = -1200$		0	0	5	0

$\Delta_j \geq 0$ (optimum solution)

$$\boxed{\begin{array}{l} \max z^* = -1200 \\ \min z = 1200 \\ x_1 = 80, x_2 = 120, S_2 = 60 \end{array}}$$

Ques – $\text{Max } z = 4x_1 + 5x_2 - 3x_3$

Subject to: $x_1 + x_2 + x_3 = 10$

$x_1 - x_2 \geq 1$

$2x_1 + 3x_2 + x_3 \leq 40$

$x_1, x_2, x_3 \geq 0$

Sol: $\text{Max } z = 4x_1 + 5x_2 - 3x_3 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to: $x_1 + x_2 + x_3 + A_1 = 10$

$x_1 - x_2 - S_1 + A_2 = 1$

$2x_1 + 3x_2 + x_3 + S_2 = 40$

$x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$

	C_j		4	5	-3	0	0	-M	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	X_B/x_j
A_1	-M	10	1	1	1	0	0	1	0	10
S_2	-M	1	$\boxed{1}$	-1	0	-1	0	0	1	1 → min ratio
S_2	0	40	2	3	1	1	1	0	0	20
Δ_j	$z = -11M$		$-2M+4$	-5	$-M+3$	M	0	0	0	

$\Delta_1 = -2M+4$

$\Delta_2 = -5$

$\Delta_3 = -M+3$

$\Delta_4 = M$

$\Delta_5 = 0$

$\Delta_6 = 0$

$\Delta_7 = 0$

$R_1 \to R_1 - R_2 \quad , \quad R_3 \to R_3 - 2R_2$

	C_j		4	5	-3	0	0	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	X_B/X_j
A_1	-M	9	0	$\boxed{2}$	1	1	0	1	$\frac{9}{2} = 4.5 \to$ min
x_1	4	1	1	-1	0	-1	0	0	-1 (neglect)
S_2	0	38	0	5	1	2	1	0	$\frac{38}{5} = 7.6$
Δ_j	$Z=-9M+4$		0	-2M-9	-M+3	-M-4	0	0	

$\Delta_1 = 4-4 = 0$

$\Delta_2 = -2M-9$

$\Delta_3 = -M+3$

$\Delta_4 = -M-4$

$\Delta_5 = 0$

$\Delta_6 = 0$

$R_1 \to R_1/2 \quad , \quad R_2 \to R_2 + R_1 \quad , \quad R_3 \to R_3 - 5R_1$

	C_j		4	5	-3	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2
x_2	5	9/2	0	1	1/2	1/2	0
x_1	4	11/2	1	0	1/2	-1/2	0
S_2	0	31/2	0	0	-3/2	-1/2	1
Δ_j	$Z = 89/2$		0	0	15/2	1/2	0

$\Delta_1 = 0$

$\Delta_2 = 0$

$\Delta_3 = \frac{5}{2} + 2 + 3 = \frac{15}{2}$

$\Delta_4 = \frac{5}{2} - 2 = \frac{1}{2}$

$\Delta_5 = 0$

$\Delta_j \geq 0$ (Optimum solution)

$$\boxed{\begin{array}{l} \max Z = 89/2 \\ x_1 = \frac{11}{2}, \; x_2 = \frac{9}{2}, \; S_2 = \frac{31}{2} \end{array}}$$

Ques – Min $z = x_1 + x_2 + 3x_3$

Subject to: $3x_1 + 2x_2 + x_3 \le 3$

$2x_1 + x_2 + 2x_3 \ge 3$

$x_1, x_2, x_3 \ge 0$

Sol: Converting minimization into maximization

Max $z^* = (-z) = -x_1 - x_2 - 3x_3 + 0S_1 + 0S_2 - MA_1$

Subject to: $3x_1 + 2x_2 + x_3 + S_1 = 3$

$2x_1 + x_2 + 2x_3 - S_2 + A_1 = 3$

	C_j		-1	-1	-3	0	0	$-M$	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	X_B/X_j
S_1	0	3	$\boxed{3}$	2	1	1	0	0	$\frac{3}{3} = 1 \rightarrow$ min
A_1	$-M$	3	2	1	2	0	-1	1	$\frac{3}{2} = 1.5$
Δ_j	$Z^* = -3M$		$-2M+1$	$-M+1$	$-2M+3$	0	M	0	

$\Delta_1 = -2M+1$

$\Delta_2 = -M+1$

$\Delta_3 = -2M+3$

$\Delta_4 = 0$

$\Delta_5 = M$

$\Delta_6 = 0$

$R_1 \rightarrow R_1/3, \quad R_2 \rightarrow R_2 - 2R_1$

	C_j		-1	-1	-3	0	0	$-M$	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	X_B/X_j
x_1	-1	1	1	$2/3$	$1/3$	$1/3$	0	0	$1 \times \frac{3}{1} = 3$
A_1	$-M$	1	0	$-1/3$	$\boxed{4/3}$	$-2/3$	-1	1	$1 \times \frac{3}{4} = 0.7 \rightarrow$ min
Δ_j	$Z^* = -M-1$		0	$\frac{M}{3} + \frac{1}{3}$	$\frac{-4M}{3} + \frac{8}{3}$	$\frac{1}{3}$	M	0	

$\Delta_1 = -1 + 1 = 0$

$\Delta_2 = -\frac{2}{3} + \frac{M}{3} + 1 = \frac{M}{3} + \frac{1}{3}$

$\Delta_3 = \frac{-1}{3} - \frac{4M}{3} + 3 = \frac{-4M}{3} + \frac{8}{3}$

$\Delta_4 = \frac{-1}{3} + \frac{2}{3} = \frac{1}{3}$

$\Delta_5 = M$

$\Delta_6 = -M + M \neq 0$

$R_2 \to R_2 \times \frac{3}{4}$, $R_1 \to R_1 - \frac{1}{3} R_2$

	C_j		-1	-1	-3	0	0
B.V.	C_B	x_B	x_1	x_2	x_3	S_1	S_2
x_1	-1	3/4	1	3/4	0	1/2	1/4
x_3	-3	3/4	0	-1/4	1	-1/2	-3/4
Δ_j	$z^* = -3$		0	1	0	0	2

$\Delta_1 = -1 + 1 = 0$

$\Delta_2 = -\frac{3}{4} + \frac{3}{4} + 1 = 1$

$\Delta_3 = -3 + 3 = 0$

$\Delta_4 = -\frac{1}{2} + \frac{1}{2} = 0$

$\Delta_5 = -\frac{1}{4} + \frac{9}{4} = \frac{8}{4} = 2$

$\max z^* = -3$

$$\boxed{\begin{array}{l} \min z = 3 \\ x_1 = \frac{3}{4}, \quad x_3 = \frac{3}{4} \end{array}}$$

* Two-Phase method –

Ques- Solve the following L.P.P. by two-phase method–

$\min z = x_1 + x_2$

Subject to: $2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

Sol: $\max z^* = (-z) = -x_1 - x_2$

Subject to: $2x_1 + x_2 - S_1 + A_1 = 4$

$x_1 + 7x_2 - S_2 + A_2 = 7$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

where $S_1, S_2 \to$ Surplus variable

$A_1, A_2 \to$ Artificial variable

$x_1 = x_2 = S_1 = S_2 = 0$

$A_1 = 4$

$A_2 = 7$

<u>Phase I</u> – Assign a cost (–1) to Artificial variable and a cost (0) to all other variables.

$$z' = 0x_1 + 0x_2 + 0S_1 + 0S_2 - A_1 - A_2$$

where z' is a new objective function.

	C_j		0	0	0	0	–1	–1	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	A_1	A_2	X_B/x_j
A_1	–1	4	2	1	–1	0	1	0	4
A_2	–1	7	1	$\boxed{7}$	0	–1	0	1	1 → min ratio
Δ_j	$z' = -11$		–3	–8	1	1	0	0	

most negative

$\Delta_1 = -2-1 = -3$

$\Delta_2 = -1-7 = -8$

$\Delta_3 = 1$

$\Delta_4 = 1$

$\Delta_5 = 0$

$\Delta_6 = 0$

$R_2 \to R_2/7$, $R_1 \to R_1 - R_2$

	C_j		0	0	0	0	–1	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	A_1	X_B/x_j
A_1	–1	3	$\boxed{13/7}$	0	–1	1/7	1	$\frac{21}{13} = 1.$ → min
x_2	0	1	1/7	1	0	–1/7	0	7 ~~→ min~~
Δ_j	$z' = -3$		–13/7	0	1	–1/7	0	

$\Delta_1 = -13/7$

$\Delta_2 = 0$

$\Delta_3 = 1$

$\Delta_4 = -1/7$

$\Delta_5 = 0$

$R_1 \to R_1 \times \frac{7}{13}$, $R_2 \to R_2 - \frac{1}{7} R_1$

	C_j		0	0	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2
x_1	0	21/13	1	0	-7/13	1/13
x_2	0	10/13	0	1	1/13	-2/13
Δ_j	$Z'=0$		0	0	0	0

$\Delta_1 = 0$ $\qquad$ $\Delta_3 = 0$

$\Delta_2 = 0$ $\qquad$ $\Delta_4 = 0$

$$\Delta_j \geq 0$$

Phase II — $\quad Z^* = -x_1 - x_2$

	C_j		-1	-1	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2
x_1	-1	21/13	1	0	-7/13	1/13
x_2	-1	10/13	0	1	1/13	-2/13
Δ_j	$Z^*=-31/13$		0	0	6/13	1/13

$\Delta_1 = -1+1 = 0$

$\Delta_2 = 0$

$\Delta_3 = \frac{7}{13} - \frac{1}{13} = \frac{6}{13}$

$\Delta_4 = -\frac{1}{13} + \frac{2}{13} = \frac{1}{13}$

$$\Delta_j \geq 0$$

$$\boxed{\begin{array}{l} Z^* = -\frac{31}{13}, \quad Z = \frac{31}{13} \\ x_1 = \frac{21}{13}, \quad x_2 = \frac{10}{13} \end{array}}$$

Ques- Solve by Two-phase method -

$$\text{Max } z = 2x_1 - x_2 + x_3$$

Subject to: $x_1 + x_2 - 3x_3 \leq 8$

$4x_1 - x_2 + x_3 \geq 2$

$2x_1 + 3x_2 - x_3 \geq 4$

$x_1, x_2, x_3 \geq 0$

Sol: $\text{Max } z = 2x_1 - x_2 + x_3$

subject to: $x_1 + x_2 - 3x_3 + S_1 = 8$

$4x_1 - x_2 + x_3 - S_2 + A_1 = 2$

$2x_1 + 3x_2 - x_3 - S_3 + A_2 = 4$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

Initial feasable solution

$x_1 = x_2 = S_1 = S_2 = S_3 = 0$

$A_1 = 2$

$A_2 = 4$

Phase I- Assign a cost (-1) to artificial variables and a cost 0 to other variables, then the new objective function -

$$z' = 0x_1 + 0x_2 + 0x_3 + 0S_1 + 0S_2 + 0S_3 - A_1 - A_2$$

	Cj		0	0	0	0	0	0	-1	-1	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	A_1	A_2	X_B/x_j
S_1	0	8	1	1	-3	1	0	0	0	0	8
A_1	-1	2	$\boxed{4}$	-1	1	0	-1	0	1	0	1/2 → min
A_2	-1	4	2	3	-1	0	0	-1	0	1	2
Δ_j	$z' = -6$		-6	-2	0	0	1	1	0	0	

$\Delta_1 = -6$

$\Delta_2 = 1 - 3 = -2$

$\Delta_3 = 0$

$\Delta_4 = 0$

$\Delta_5 = 1$

$\Delta_6 = 1$

$\Delta_7 = 0$

$\Delta_8 = 0$

$R_2 \to \frac{R_2}{4}$, $R_1 \to R_1 - R_2$, $R_3 \to R_3 - 2R_2$

	C_j		0	0	0	0	0	0	-1	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	A_2	X_B/X_j
S_1	0	15/2	0	5/4	-13/4	1	1/4	0	0	$\frac{15}{2} \times \frac{4}{5} = 6$
x_1	0	1/2	1	-1/4	1/4	0	-1/4	0	0	-ve
A_2	-1	3	0	$\boxed{7/2}$	-3/2	0	1/2	-1	1	$3 \times \frac{2}{7} = 6/7 \to$ min
Δ_j	$z' = -3$		0	-7/2	3/2	0	-1/2	1	0	

$\Delta_1 = 0$

$\Delta_2 = -7/2$

$\Delta_3 = 3/2$

$\Delta_4 = 0$

$\Delta_5 = -1/2$

$\Delta_6 = 1$

$\Delta_7 = 0$

$R_3 \to R_3 \times \frac{2}{7}$, $R_2 \to R_2 + \frac{1}{4} R_3$, $R_1 \to R_1 - \frac{5}{4} R_3$

	C_j		0	0	0	0	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
S_1	0	45/7	0	0	-19/7	1	1/14	5/14
x_1	0	5/7	1	0	1/7	0	-3/14	-1/14
x_2	0	6/7	0	1	-3/7	0	1/7	-2/7
Δ_j	$z' = 0$		0	0	0	0	0	0

<u>Phase II</u> $z = 2x_1 - x_2 + x_3$

	C_j		2	-1	1	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/X_j
S_1	0	45/7	0	0	-19/7	1	1/14	5/14	$\frac{45}{7} \times \frac{14}{1} = 90$
x_1	2	5/7	1	0	1/7	0	-3/14	-1/14	-ve
x_2	-1	6/7	0	1	-3/7	0	$\boxed{1/7}$	-2/7	$\frac{6}{7} \times \frac{7}{1} = 6 \to$ min
Δ_j	$z = 4/7$		0	0	-2/7	0	-4/7	1/7	

$\Delta_1 = 7 - 2 - 2 = 0$

$\Delta_2 = 0$

$\Delta_3 = \frac{2}{7} + \frac{3}{7} - 1 = -\frac{2}{7}$

$\Delta_4 = 0$

$\Delta_5 = -6/14 - 1/7 = -4/7$

$\Delta_6 = -2/14 + 2/7 = 1/7$

$R_3 \to 7R_3$, $R_1 \to R_1 - R_2 \times \frac{1}{14}$, $R_2 \to R_2 + \frac{3}{14} R_3$

	C_j		2	-1	1	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/x_j
S_1	0	6	0	-1/2	-5/2	1	0	1/2	-ve
x_1	2	2	1	3/2	-1/2	0	0	-1/2	-ve
S_2	0	6	0	7	-3	0	1	-2	-ve
Δ_j	$z=4$		0	4	-2	0	0	-1	

$\Delta_1 = 2-2=0$

$\Delta_2 = 3+1=4$

$\Delta_3 = -1+1=-2$

$\Delta_4 = 0$

$\Delta_5 = 0$

$\Delta_6 = -1$

★ Graphical method :-

Ques- Mini $z = 2x_1 + 3x_2$

Subject to: $x_1 + x_2 \geq 6$

$7x_1 + x_2 \geq 14$

$x_1, x_2 \geq 0$

Sol:

$x_1 + x_2 = 6$ — (1)

$7x_1 + x_2 = 14$ — (2)

$x_1, x_2 \geq 0$

Put $x_1 = 0$ in eq (1)

$x_2 = 6$, Point $(0,6)$

Put $x_2 = 0$ in eq (1)

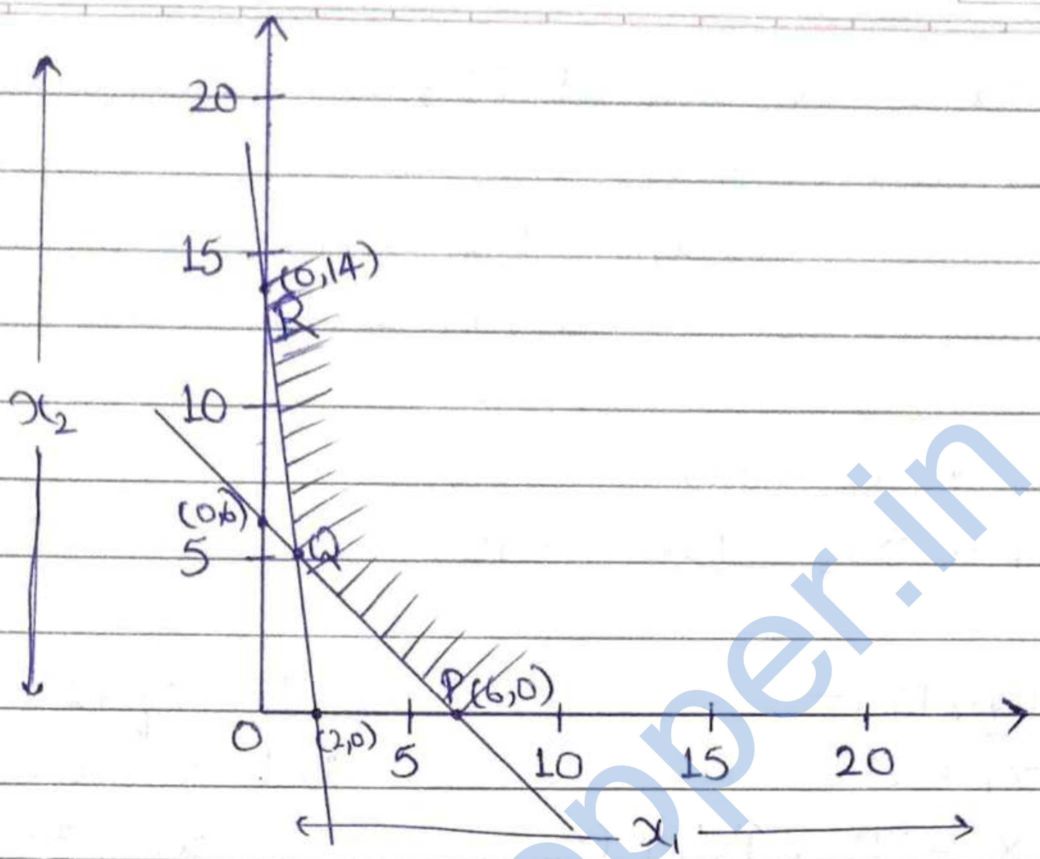
$x_1 = 6$, Point $(6,0)$

Put $x_1 = 0$ in eq (2)

$x_2 = 14$, Point $(0,14)$

Put $x_2 = 0$ in eq (2)

$x_1 = 2$, Point $(2,0)$



$$7x_1 + x_2 = 14$$
$$x_1 + x_2 = 6$$
$$- \quad - \quad -$$

$$6x_1 = 8$$
$$x_1 = 4/3$$

$$x_2 = 6 - x_1$$
$$x_2 = 6 - \frac{4}{3}$$
$$x_2 = \frac{14}{3}$$

The co-ordinate of Q is $\left(\frac{4}{3}, \frac{14}{3}\right)$

Corner Point	min $z = 2x_1 + 3x_2$	
P(6,0)	$2 \times 6 + 0$	12
Q (4/3, 14/3)	$2 \times \frac{4}{3} + 3 \times \frac{14}{3}$	50/3
R (0,14)	0+42	42

$$\min z = 12$$
$$x_1 = 6$$
$$x_2 = 0$$

Ques – Max $z = 3x_1 + 4x_2$

Subject to: $4x_1 + x_2 \geq 80$

$2x_1 + 5x_2 \leq 180$

$x_1, x_2 \geq 0$

Sol:

$4x_1 + x_2 = 80 \quad —(1)$

$2x_1 + 5x_2 = 180 \quad —(2)$

In eq(1) –

Put $x_1 = 0$

$x_2 = 80$, Point $(0, 80)$

Put $x_2 = 0$

$x_1 = 20$, Point $(20, 0)$

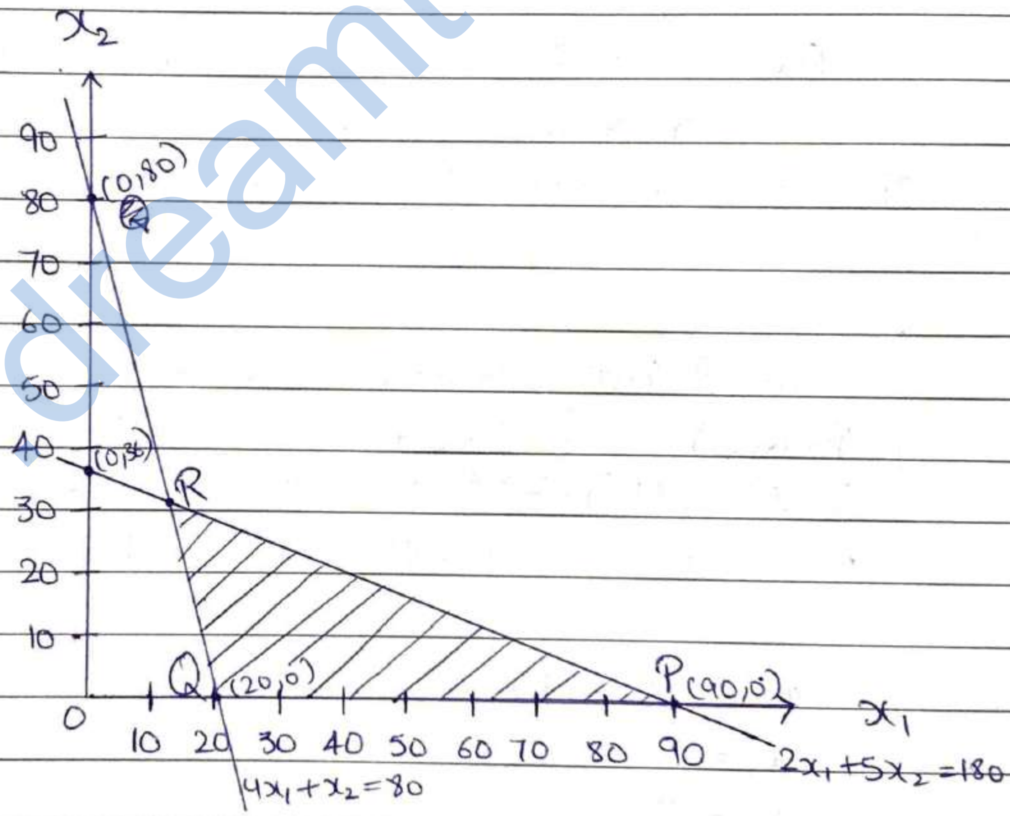
In eq(2) –

Put $x_1 = 0$

$x_2 = 36$, Point $(0, 36)$

Put $x_2 = 0$

$x_1 = 90$, Point $(90, 0)$



$4\cancel{x_1} + x_2 = 80$

$\cancel{4}x_1 + 10x_2 = 360$

$- \quad - \quad -$

$-8x_2 = -280$

$\boxed{x_2 = 35}$

Corner Point	max $z = 3x_1 + 4x_2$
$P(90,0)$	270 → max
$Q(20,0)$	60
$R(5/2, 35)$	$3\times\frac{5}{2} + 4\times 35 = 295/2$

max $z = 270$

$x_1 = 90$, $x_2 = 0$

Ques- $x_1 - 2x_2 \le 1$

$x_1 + 2x_2 \ge 3$

$x_1, x_2 \ge 0$

Sol: $x_1 - 2x_2 = 1$ — (1)

$x_1 + 2x_2 = 3$ — (2)

In eq (1)

$x_1 = 0$

$x_2 = -1/2$ Point $(0, -1/2)$

Put $x_2 = 0$

$x_1 = 1$ Point $(1,0)$

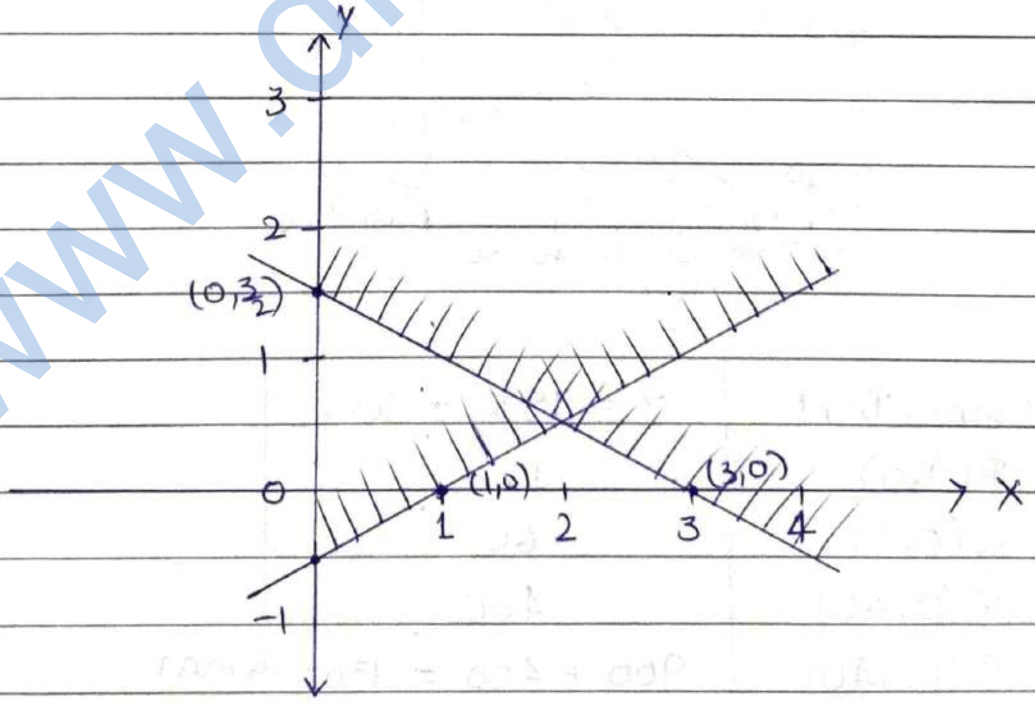
In eq (2)

Put $x_1 = 0$

$x_2 = 3/2$ Point $(0, 3/2)$

Put $x_2 = 0$

$x_1 = 3$ Point $(3,0)$



Ques - Solve by graphical method the following L.P.P. -

$$\text{Max } z = 15x_1 + 10x_2$$

Subject to: $4x_1 + 6x_2 \le 36$

$3x_1 + 0x_2 \le 180$

$0x_1 + 5x_2 \le 200$

$x_1, x_2 \ge 0$

Sol:

$4x_1 + 6x_2 = 36$ — (1)

$3x_1 + 0x_2 = 180$ — (2)

$0x_1 + 5x_2 = 200$ — (3)

In eq (1)

Put $x_1 = 0$

$x_2 = 6$, Point (0,6)

Put $x_2 = 0$

$x_1 = 9$, Point (9,0)

In eq (2)

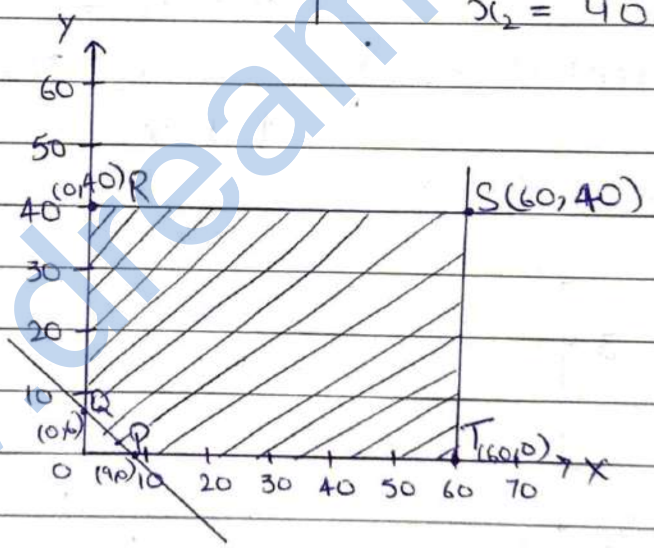
$3x_1 = 180$

$x_1 = 60$, Point (60,0)

In eq (3)

$5x_2 = 200$

$x_2 = 40$, Point (0,40)



Corner Point	$z = 15x_1 + 10x_2$
P(9,0)	135
Q(0,6)	60
R(0,40)	400
S(60,40)	900 + 400 = 1300 → max
T(60,0)	900

$\max z = 1300$

$x_1 = 60$, $x_2 = 40$.

Ques- max $z = 8x_1 + 5x_2$

subject to: $2x_1 + x_2 \geq 8$

$x_1 + x_2 \geq 6$

$x_1 + 3x_2 \geq 9$

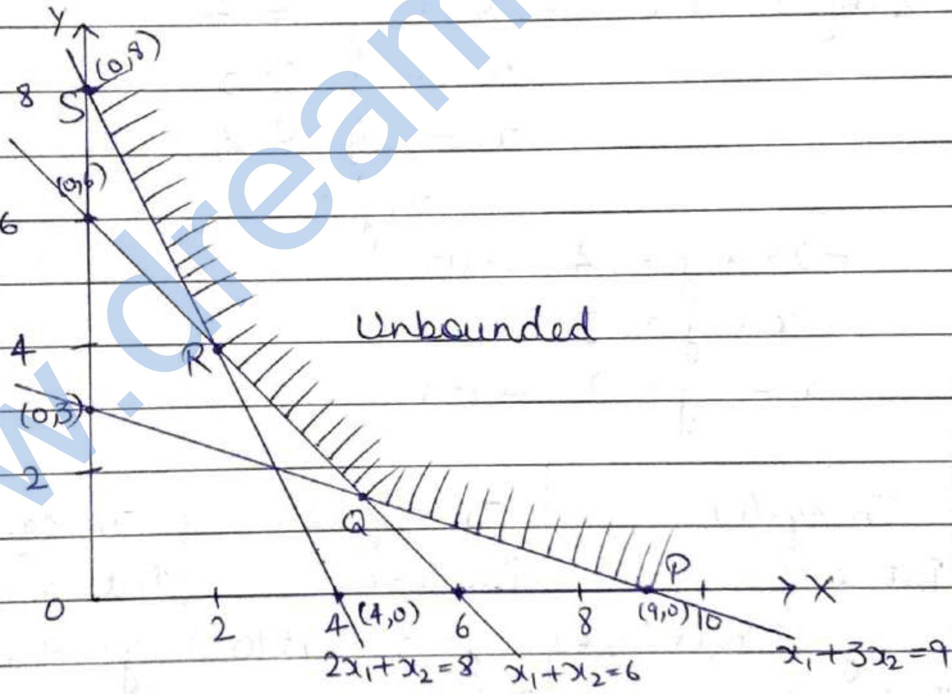
$x_1, x_2 \geq 0$

Sol: $2x_1 + x_2 = 8$ — (1)

$x_1 + x_2 = 6$ — (2)

$x_1 + 3x_2 = 9$ — (3)

In eq (1)	In eq (2)	In eq (3)
Put $x_1 = 0$	Put $x_1 = 0$	Put $x_1 = 0$
$x_2 = 8$, Point $(0,8)$	$x_2 = 6$, Point $(0,6)$	$x_2 = 3$, Point $(0,3)$
Put $x_2 = 0$	Put $x_2 = 0$	Put $x_2 = 0$
$x_1 = 4$, Point $(4,0)$	$x_1 = 6$, Point $(6,0)$	$x_1 = 9$, Point $(9,0)$



$$\begin{aligned} \cancel{x_1} + x_2 &= 6 \\ \cancel{x_1} + 3x_2 &= 9 \\ - \quad - \quad &\;\; - \\ \hline -2x_2 &= -3 \\ x_2 &= \frac{3}{2} \end{aligned}$$

$$x_1 + x_2 = 6$$

$$x_1 = 6 - \frac{3}{2}$$

$$x_1 = \frac{9}{2} \Rightarrow Q\left(\frac{3}{2}, \frac{9}{2}\right)$$

$$2x_1 + \cancel{x_2} = 8$$
$$\underline{\begin{array}{c} - \\ x_1 \end{array} + \begin{array}{c} - \\ \cancel{x_2} \end{array} = \begin{array}{c} - \\ 6 \end{array}}$$
$$x_1 = 2$$

$$x_1 + x_2 = 6$$
$$2 + x_2 = 6$$
$$x_2 = 4 \Rightarrow R(2,4)$$

Corner Point	$z = 8x_1 + 5x_2$
$P(9,0)$	$72 \longrightarrow$ max
$Q(9/2, 3/2)$	$36 + \frac{15}{2} = 87/2$
$R(2,4)$	$16 + 20 = 36$
$S(0,8)$	40

$$\max z = 72$$
$$x_1 = 9, \; x_2 = 0.$$

Ques– $\min z = 3x + 5y$

subject to: $-2x + y \le 4$

$x + y \ge 3$

$x - 2y \le 2$

$x, y \ge 0$

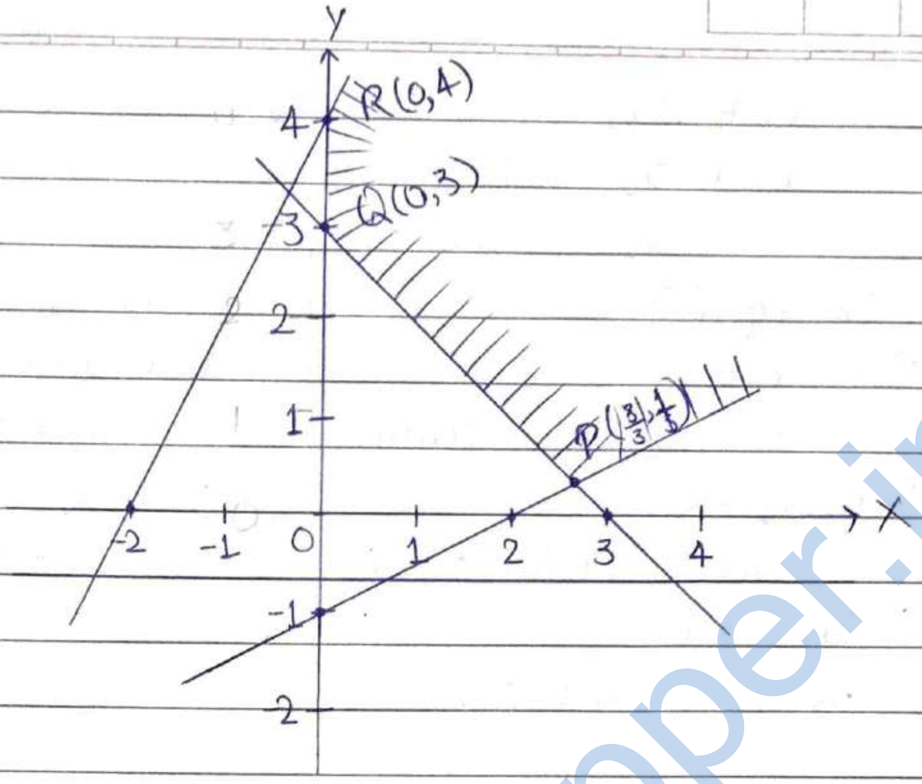
Sol:

$-2x + y = 4$ — (1)

$x + y = 3$ — (2)

$x - 2y = 2$ — (3)

In eq (1)	In eq (2)	In eq (3)
Put $x=0$	Put $x=0$	Put $x=0$
$y=4$, Point $(0,4)$	$y=3$, Point $(0,3)$	$y=-1$, Point $(0,-1)$
Put $y=0$	Put $y=0$	Put $y=0$
$x=-2$, Point $(-2,0)$	$x=3$, Point $(3,0)$	$x=2$, Point $(2,0)$



$$x + y = 3$$
$$x - 2y = 2$$
$$- \quad + \quad -$$

$$3y = 1$$
$$y = 1/3$$

$$x + y = 3$$
$$x = 3 - \frac{1}{3}$$
$$x = \frac{8}{3} \Rightarrow P\left(\frac{8}{3}, \frac{1}{3}\right)$$

Corner Point	$Z = 3x + 5y$
$P(8/3, 1/3)$	$29/3 = 9.6 \rightarrow$ min
$Q(0,3)$	15
$R(0,4)$	20

$$\min Z = 9.6$$
$$x = \frac{8}{3},\ y = \frac{1}{3}.$$

Ques- Max $Z = 3x_1 + 5x_2$

Subject to: $x_1 + 2x_2 \leq 20$

$x_1 + x_2 \leq 15$

$x_2 \leq 6$

$x_1, x_2 \geq 0$

Sol:

$$x_1 + 2x_2 = 20 \quad \text{— (1)}$$
$$x_1 + x_2 = 15 \quad \text{— (2)}$$
$$x_2 = 6 \quad \text{— (3)}$$

In eq (1)

Put $x_1 = 0$

$x_2 = 10$, Point $(0,10)$

Put $x_2 = 0$

$x_1 = 20$, Point $(20,0)$

In eq (3)

$x_2 = 6$, Point $(0,6)$

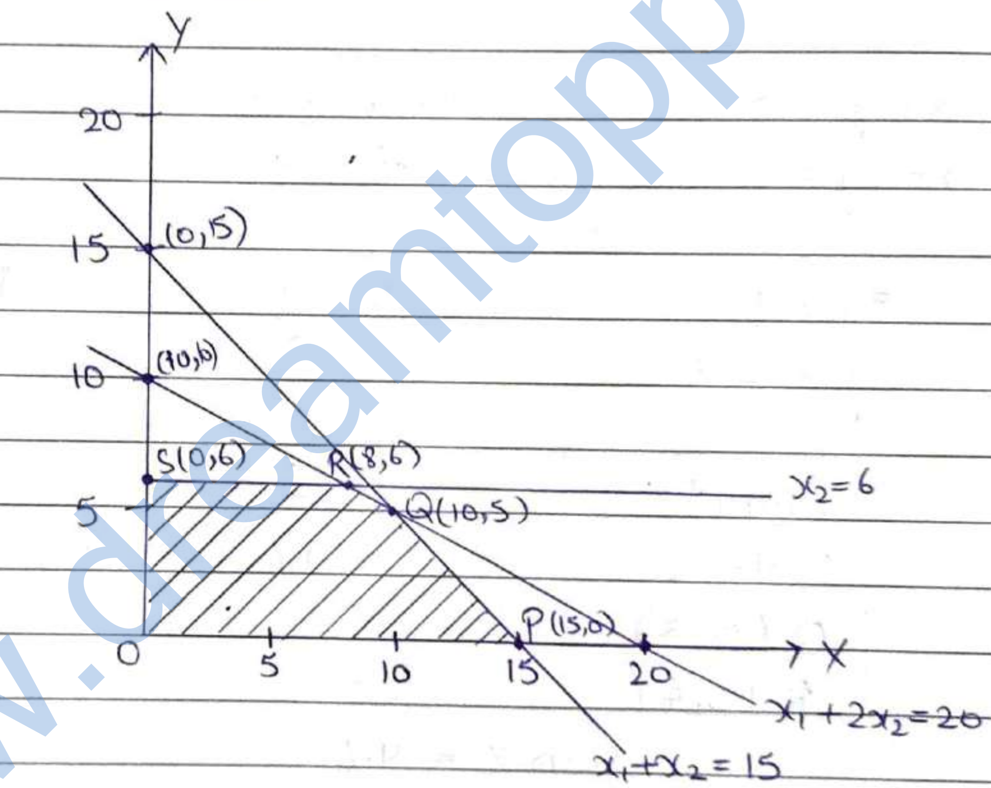
In eq (2)

Put $x_1 = 0$

$x_2 = 15$, Point $(0,15)$

Put $x_2 = 0$

$x_1 = 15$, Point $(15,0)$



$$\cancel{x_1} + 2x_2 = 20$$
$$\cancel{x_1} + x_2 = 15$$
$$- \quad - \qquad -$$
$$x_2 = 5$$

$$x_1 + x_2 = 15$$
$$x_1 = 15-5$$
$$x_1 = 10$$
$$\Rightarrow Q(10,5)$$

$$x_2 = 6$$
$$x_1 + 2x_2 = 20$$
$$x_1 = 20-12$$
$$x_1 = 8 \qquad \Rightarrow R(8,6)$$

Corner Point	$Z = 3x_1 + 5x_2$
$P(15, 0)$	45
$Q(10, 5)$	$30 + 25 = 55$ → max
$R(8, 6)$	$24 + 30 = 54$
$S(0, 10)$	50

$$\max Z = 55$$

$$x_1 = 10, \quad x_2 = 5.$$

* Duality in L.P.P. :-

Ques- $\min Z = 10x_1 + 20x_2$

subject to:
$$3x_1 + 2x_2 \geq 18$$
$$x_1 + 3x_2 \geq 8$$
$$2x_1 - x_2 \leq 6$$
$$x_1, x_2 \geq 0$$

Sol: Given L.P.P.

$$3x_1 + 2x_2 \geq 18$$
$$x_1 + 3x_2 \geq 8$$
$$2x_1 - x_2 \leq 6$$

Primal form -

$$3x_1 + 2x_2 \geq 18$$
$$x_1 + 3x_2 \geq 8$$
$$-2x_1 + x_2 \geq -6$$

Matrix form -

$$\begin{bmatrix} 3 & 2 \\ 1 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 18 \\ 8 \\ 6 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

$$Ax \geq b$$

$$\min Z = (10, 20)(x_1, x_2) = cx$$

Dual

$$\text{Max } Z_D = b'y$$

$$\max Z_D = [18, 8, -6]\,[y_1, y_2, y_3]$$

$$\max Z_D = 18y_1 + 8y_2 - 6y_3$$

$$y_1, y_2, y_3 \geq 0$$

$$A'y \leq c'$$

$$\begin{bmatrix} 3 & 1 & -2 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

$$3y_1 + y_2 - 2y_3 \leq 10$$

$$2y_1 + 3y_2 + y_3 \leq 20$$

$$y_1, y_2, y_3 \geq 0 \quad \text{Ans.}$$

Ques- Write the dual of the following L.P.P. —

$$\text{Max } z = 2x_1 + 3x_2 + x_3$$

Subject to: $4x_1 + 3x_2 + x_3 = 6$

$$x_1 + 2x_2 + 5x_3 = 4$$

$$x_1, x_2, x_3 \geq 0$$

Sol: Change into standard Primal form –

$$4x_1 + 3x_2 + x_3 \leq 6$$

$$-4x_1 - 3x_2 - x_3 \leq -6$$

$$x_1 + 2x_2 + 5x_3 \leq 4$$

$$-x_1 - 2x_2 - 5x_3 \leq -4$$

$$x_1, x_2, x_3 \geq 0$$

Matrix form –

$$\begin{bmatrix} 4 & 3 & 1 \\ -4 & -2 & -1 \\ 1 & 2 & 5 \\ -1 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 6 \\ -6 \\ 4 \\ -4 \end{bmatrix}$$

$$Ax \leq b$$

$$\text{Max } Z = (2\ 3\ 1)\,(x_1\ x_2\ x_3) \Rightarrow cx$$

Duality –

$$\text{Mini } Z_D = b'y$$

$$\text{Mini } Z_D = [6 \ -6 \ 4 \ -4] \, [y_1' \ y_1'' \ y_2' \ y_2'']$$

$$= 6y_1' - 6y_1'' + 4y_2' - 4y_2''$$

$$= 6(y_1' - y_1'') + 4(y_2' - y_2'')$$

$$y_1', y_1'', y_2', y_2'' \geq 0$$

$$A'y \geq c'$$

$$\begin{bmatrix} 4 & -4 & 1 & -1 \\ 3 & -3 & 2 & -2 \\ 1 & -1 & 5 & -5 \end{bmatrix} \begin{bmatrix} y_1' \\ y_1'' \\ y_2' \\ y_2'' \end{bmatrix} \geq \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$4y_1' - 4y_1'' + y_2' - y_2'' \geq 2 \Rightarrow 4(y_1' - y_1'') + (y_2' - y_2'') \geq 2$$

$$3y_1' - 3y_1'' + 2y_2' - 2y_2'' \geq 3 \Rightarrow 3(y_1' - y_1'') + 2(y_2' - y_2'') \geq 3$$

$$y_1' - y_1'' + 5y_2' - 5y_2'' \geq 1 \Rightarrow (y_1' - y_1'') + 5(y_2' - y_2'') \geq 1$$

Let $y_1 = y_1' - y_1''$

$y_2 = y_2' - y_2''$

$$4y_1 + y_2 \geq 2$$

$$3y_1 + 2y_2 \geq 3$$

$$y_1 + 5y_2 \geq 1$$

$$y_1, y_2 \geq 0$$

$\therefore$ Mini $Z = 6y_1 + 4y_2$ Ans.

Ques– Write the dual of the following L.P.P. –

$$\text{Min } Z = x_1 + x_2 + x_3$$

Subject to: $x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \leq 3$

$2x_2 - x_3 \geq 4$

$x_1, x_2, x_3 \geq 0$

Sol: Change into standard primal form –

$$x_1 - 3x_2 + 4x_3 \geq 5$$
$$-x_1 + 3x_2 - 4x_3 \geq -5$$
$$-x_1 + 2x_2 + 0x_3 \geq -3$$
$$0x_1 + 2x_2 - x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

Matrix form –

$$\begin{bmatrix} 1 & -3 & 4 \\ -1 & 3 & -4 \\ -1 & 2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 5 \\ -5 \\ -3 \\ 4 \end{bmatrix}$$

$$Ax \geq b$$

$$\text{Mini } Z = (1,1,1)(x_1, x_2, x_3) = cx$$

Duality –

$$\text{Max } Z_D = b'y$$
$$\text{Max } Z_D = [5 \ -5 \ -3 \ 4][y_1' \ y_1'' \ y_2 \ y_3]$$
$$= 5y_1' - 5y_1'' - 3y_2 + 4y_3$$
$$= 5(y_1' - y_1'') - 3y_2 + 4y_3$$
$$y_1', y_1'', y_2, y_3 \geq 0$$

$$Ay \leq c'$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ -3 & 3 & 2 & 2 \\ 4 & -4 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1' \\ y_1'' \\ y_2 \\ y_3 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$(y_1' - y_1'') - y_2 \leq 1$$
$$-3(y_1' - y_1'') + 2y_2 + 2y_3 \leq 1$$
$$4(y_1' - y_1'') - y_3 \leq 1$$

Let $(y_1' - y_1'') = y_1$

$$y_1 - y_2 \le 1$$
$$-3y_1 + 2y_2 + 2y_3 \le 1$$
$$4y_1 - y_3 \le 1$$

$\therefore$ Max $Z = 5y_1 - 3y_2 + 4y_3$ Ay.

★ What is Operations Research —

"O.R. is the art of giving bad answers to problems which otherwise have worse answers."

— T.L. Saaty

"O.R. is the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solutions to the problem."

— C.W. Churchman

★ Essential characteristics of Operations Research —

The significant features of O.R. are given below —

1- Decision making — A major premise of O.R. is that decision-making, irrespective of the situation involved, can be considered as a general systematic process.

2- Scientific approach – O.R. employs scientific methods for the purpose of solving problems. It is a formalised process of reasoning.

3- Objective – O.R. attempts to locate the best or optimal solution to the problem under consideration

4- Inter- disciplinary Team approach – O.R. is inter- disciplinary in nature and requires a team approach to a solution of the problem. Managerial problems have economic, physical, biological and engineering aspects. This requires a blend of people with expertise in the areas of mathematics, statistics, engineering, economics, management, computer science and so on.

5- Digital computer – Use of digital computer has become an integral part of O.R.'s approach to decision - making. The computer may be required due to complexity of the model, volume of data required and the computations to be made.

* Advantages and Limitations of a Model :-

Advantages of a model –

1. Through a model, the position under consideration becomes controllable.
2. It help in finding evenues for few research and improvements in a system.
3. It indicates the limitations and scope of an activity.

4. It provide some logical and systematic approach to the problem.

Limitations of a model –

1. Models are only an attempt in understanding operations and should never be considered as absolute in any sense.
2. Validity of any model with regard to corresponding operation can only be verified by carrying the experiment and relevant data characteristics.

* Limitations of Operations Research :–

1- Mathematical models which are essences of O.R. do not take into account quantitative factors which are quite real. All influencing factors which cannot be quantified find no place in mathematical models
2- Mathematical models are applicable to only specific categories of problems.
3- Being the new field there is resistance from the employees to the new proposals.
4- Management, who has to implement the advised proposals, may itself offer a lot of resistance due to conventional thinking.
5- Young enthusiasts, overtaken by its advantages and exactness generally forget that O.R. is meant for men and not that men are meant for it.

* Techniques Used in Operations Research –

The following techniques used in operations research –

1- Linear Programming – It is used in the solution of problems concerned with assignment of personal, blending of materials, distribution and transportation and investment properties.

2- Dynamic programming – It is used in such area as planning, advertising expenditures distributing sales effort and production scheduling etc.

3- Queuing theory – It is used in solving problem concern with traffic, servicing machines subject to break down, air traffic scheduling, production scheduling, hospital operations, determining optimum number of replacement for a group of machines.

4- Inventory theory – In determining when and how much a production or purchase.

5- Game theory – The primary objective of game theory is to develop rational criteria for selecting a strategy.

6- Simulations – The technique of simulation is an important tools of the designer in stimulating airplane flight in a wind tunnel, simulating lines of communication with an organisation chart. With the advent of the high speed digital computer with which to conduct simulated experiments, this technique has become experiment-al arm of researches.

★ Conversion of a L.P.P. into Standard Form :-

Step 1 - Change the linear constraints of inequality type into equations :

This is done with the help of 'slack' or 'surplus' variables. These non-negative variables are added to (or subtracted from) the left hand side of each such constraint. These new variables are added if the constraint is ($\leq$) and are called 'slack variables'. On the other hand, these new variables are subtracted if the constraint is ($\geq$) and are called 'surplus variables'.

For example-

1.) The linear constraint $3x_1 + 4x_2 \leq 20$ is converted to an equation with slack variables S_1 (or x_3) as -

$$3x_1 + 4x_2 + S_1 = 20, \quad \text{where } S_1 \geq 0$$

$$\{\text{or } 3x_1 + 4x_2 + x_3 = 20, \quad \text{where } x_3 \geq 0\}$$

2.) The linear constraint $3x_1 + 7x_2 \geq 28$ is converted to an equation with surplus variable S_2 {or x_4}.

$$3x_1 + 7x_2 - S_2 = 28, \quad \text{where } S_2 \geq 0$$

$$\{\text{or } 3x_1 + 7x_2 - x_4 = 28, \quad \text{where } x_4 \geq 0\}$$

In the objective function the coefficient of slack and surplus variables are shown as zero.

The objective function $Z = 2x_1 + 3x_2$ is written as

$$Z = 2x_1 + 3x_2 + 0S_1 + 0S_2$$

$$\{\text{or } Z = 2x_1 + 3x_2 + 0x_3 + 0x_4\}$$

Hence, the cost of slack or surplus variable is taking zero in objective function.

Step 2 - Make the right ~~had~~ hand element of each constraint non-negative.

This is done by multiplying both sides of the resulting constraint by (-1).

For example–

Consider the constraint $2x_1 - 3x_2 + 4x_3 = -17$

This constraint is changed (in standard form) to

$$-(2x_1 - 3x_2 + 4x_3) = (-1)(-17)$$
$$-2x_1 + 3x_2 - 4x_3 = 17$$

Step 3– Make the unrestricted variables as non-negative:
This is done by replacing the unrestricted variable by a difference of two non-negative variables. For example, if x_2 is unrestricted in sign, it can be replaced by –

$$(x_2' - x_2''), \text{ where } x_2' \geq 0, x_2'' \geq 0$$
$$\text{by } (x_3 - x_4), \text{ where } x_3 \geq 0, x_4 \geq 0$$
$$\text{by } (y_1 - y_2), \text{ where } y_1 \geq 0, y_2 \geq 0$$

Step 4– Convert the objective function in maximization form:
This is done by changing the sign of the objective function.

For example–

Minimize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ is equivalent to the expression

Minimize $(-Z) = -c_1x_1 - c_2x_2 - \ldots - c_nx_n$

After going through the steps 1 to 4 a given L.P.P. in standard form as –

Optimize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n + 0S_1 + \ldots + 0S_m$

Subject to
$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \pm S_1 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \pm S_2 = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \pm S_i = b_i$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + S_m = b_m$$

and $x_1, x_2, x_3, \dots, x_n, S_1, \dots S_m \geq 0$

where $b_1, b_2, \dots, b_m \geq 0$

Since standard form involves only equations, we can write it in matrix form as follows –

Optimize $Z = CX$

Such that $AX = b$

and $X \geq 0$

Assignment on Linear Programming Problem

Ques 1: Solve the following L.P.P. by simplex method :-

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3$$

subject to : $2x_1 + x_2 \le 18$

$$2x_2 + 5x_3 \le 18$$

$$3x_1 + 2x_2 + 4x_3 \le 25$$

$$x_1, x_2, x_3 \ge 0$$

Sol:

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3 + 0S_1 + 0S_2 + 0S_3$$

subject to : $2x_1 + x_2 + 0x_3 + S_1 = 18$

$$2x_2 + 5x_3 + S_2 = 18$$

$$3x_1 + 2x_2 + 4x_3 + S_3 = 25$$

$$x_1, x_2, x_3, S_1, S_2, S_3 \ge 0$$

$S_1, S_2, S_3 \rightarrow$ Slack variables

Incoming vector

	C_j		3	5	4	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/X_j
S_1	0	18	2	1	0	1	0	0	18
Outgoing vector ← S_2	0	18	0	$\boxed{2}$	5	0	1	0	9 → min ratio
S_3	0	25	3	2	4	0	0	1	$25/2 = 12.5$
Δ_j	$Z = 0$		-3	-5	-4	0	0	0	

key element

most negative

$\Delta_1 = -3$ $\quad\quad \Delta_4 = 0$

$\Delta_2 = -5$ $\quad\quad \Delta_5 = 0$

$\Delta_3 = -4$ $\quad\quad \Delta_6 = 0$

$$R_2 \rightarrow \frac{R_2}{2}, \quad R_1 \rightarrow R_1 - R_2, \quad R_3 \rightarrow R_3 - 2R_2$$

	C_j		3	5	4	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B/X_j
S_1	0	9	2	0	$-5/2$	1	$-1/2$	0	$9/2 = 4.5$
x_2	5	9	0	1	$5/2$	0	$1/2$	0	∞ (neglect)
Outgoing vector ← S_3	0	7	$\boxed{3}$	0	-1	0	-1	1	$7/3 = 2.3$ → min ratio
Δ_j	$Z = 45$		-3	0	$17/2$	0	$5/2$	0	

most negative

$\Delta_1 = 0 - 3 = -3$ $\qquad\qquad$ $\Delta_4 = 0$

$\Delta_2 = 5 - 5 = 0$ $\qquad\qquad$ $\Delta_5 = 5/2$

$\Delta_3 = 25/2 - 4 = 17/2$ $\qquad\qquad$ $\Delta_6 = 0$

$R_3 \rightarrow \frac{R_3}{3}$, $R_1 \rightarrow R_1 - 2R_3$

	C_j		3	5	4	0	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
S_1	0	13/3	0	0	-11/6	1	1/6	-2/3
x_2	5	9	0	1	5/2	0	1/2	0
x_1	3	7/3	1	0	-1/3	0	-1/3	1/3
Δ_j	$Z = 52$		0	0	15/2	0	3/2	1

$\Delta_1 = 3 - 3 = 0$ $\qquad\qquad$ $\Delta_4 = 0$

$\Delta_2 = 5 - 5 = 0$ $\qquad\qquad$ $\Delta_5 = 5/2 - 1 = 3/2$

$\Delta_3 = \frac{25}{2} - 5 = \frac{15}{2}$ $\qquad\qquad$ $\Delta_6 = 1$

$\Delta_j \geq 0$ (Optimum solution)

$$\boxed{\begin{array}{l} \text{Max } Z = 52 \\ x_1 = \frac{7}{3},\ x_2 = 9,\ x_3 = 0,\ S_1 = \frac{13}{3} \end{array}}$$

Ques2: Solve the following L.P.P. by Big M method-

$$\text{Min } Z = 4x_1 + 3x_2$$

subject to: $2x_1 + x_2 \geq 10$

$-3x_1 + 2x_2 \leq 6$

$x_1 + x_2 \geq 6$

$x_1, x_2 \geq 0$

Sol: Converting the minimization into maximization-

$$\text{Max } Z^* = (-Z)$$

$$\text{Max } Z^* = -4x_1 - 3x_2 + 0S_1 + 0S_2 + 0S_3 - MA_1 - MA_2$$

subject to: $2x_1 + x_2 - S_1 + A_1 = 10$

$-3x_1 + 2x_2 + S_2 = 6$

$x_1 + x_2 - S_3 + A_2 = 6$

$x_1, x_2, S_1, S_2, S_3, A_1, A_2 \geq 0$

where $S_1, S_3 \rightarrow$ Surplus variable

$S_2 \rightarrow$ Slack variable

$A_1, A_2 \rightarrow$ Artificial variables

	C_j		-4	-3	0	0	0	-M	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_1	A_2	X_B/x_j
Outgoing vector ← A_1	-M	10	$\boxed{2}$	1	-1	0	0	1	0	5 → min ratio
S_2	0	6	-3	2	0	1	0	0	0	-ve
A_2	-M	6	1	1	0	0	-1	0	1	6
Δ_j	$Z^* = -16M$		-3M+4 ↓ most negative	-2M+3	M	0	M	0	0	

$\Delta_1 = -3M + 4$

$\Delta_2 = -2M + 3$

$\Delta_3 = M$

$\Delta_4 = 0$

$\Delta_5 = M$

$\Delta_6 = 0$

$\Delta_7 = 0$

$R_1 \rightarrow \frac{R_1}{2}$, $R_2 \rightarrow R_2 + 3R_1$, $R_3 \rightarrow R_3 - R_1$

	C_j		-4	-3	0	0	0	-M	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3	A_2	X_B/x_j
x_1	-4	5	1	1/2	-1/2	0	0	0	10
S_2	0	21	0	7/2	-3/2	1	0	0	6
Outgoing vector ← A_2	-M	1	0	$\boxed{1/2}$	1/2	0	-1	1	2 → min ratio
Δ_j	$Z^* = -M-20$		0	$1-\frac{M}{2}$ ↓ most negative	$2-\frac{M}{2}$	0	M	0	

$\Delta_1 = 0$

$\Delta_2 = \frac{-4}{2} - \frac{M}{2} + 3 = 1 - \frac{M}{2}$

$\Delta_3 = 2 - \frac{M}{2}$

$\Delta_4 = 0$

$\Delta_5 = M$

$\Delta_6 = -M + M = 0$

$R_3 \to R_3 \times 2$, $R_2 \to R_2 - \frac{7}{2}R_3$, $R_1 \to R_1 - \frac{1}{2}R_3$

	C_j		-4	-3	0	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	S_3
x_1	-4	4	1	0	-1	0	1
S_2	0	14	0	0	-5	1	7
x_2	-3	2	0	1	1	0	-2
Δj	$Z^* = -22$		0	0	1	0	2

$\Delta j \geq 0$, the solution is optimum.

$$\boxed{\begin{array}{c} Z^* = -22 \ , \ Z = 22 \\ x_1 = 4 \ , \ x_2 = 2 \ , \ S_2 = 14 \end{array}}$$

Ques3:- Solve the following L.P.P. by simplex method –

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3$$

subject to: $2x_1 + 3x_2 \leq 8$

$2x_2 + 5x_3 \leq 10$

$3x_1 + 2x_2 + 4x_3 \leq 15$

$x_1, x_2, x_3 \geq 0$

Sol:

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3 + 0S_1 + 0S_2 + 0S_3$$

subject to: $2x_1 + 3x_2 + S_1 = 8$

$2x_2 + 5x_3 + S_2 = 10$

$3x_1 + 2x_2 + 4x_3 + S_3 = 15$

$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

where $S_1, S_2, S_3 \to$ Slack variables

	C_j		3	5	4	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B / x_j
Outgoing vector ← S_1	0	8	2	$\boxed{3}$	0	1	0	0	$\frac{8}{3} = 2.6$ → min ratio
S_2	0	10	0	2	5	0	1	0	5
S_3	0	15	3	2	4	0	0	1	$\frac{15}{2} = 7.5$
Δj	$Z = 0$		-3	-5	-4	0	0	0	

Key element

most negative

$\Delta_1 = -3$ $\qquad\qquad$ $\Delta_4 = 0$

$\Delta_2 = -5$ $\qquad\qquad$ $\Delta_5 = 0$

$\Delta_3 = -4$ $\qquad\qquad$ $\Delta_6 = 0$

$R_1 \to \frac{R_1}{3}$, $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$

			3	5	4	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B / x_j
x_2	5	8/3	2/3	1	0	1/3	0	0	∞ (neglect)
Outgoing vector ← S_2	0	14/3	-4/3	0	$\boxed{5}$	-2/3	1	0	$\frac{14}{15} = 0.9$ → min ratio
S_3	0	29/3	5/3	0	4	-4/3	0	1	$\frac{29}{12} = 2.4$
Δ_j	Z = 40/3		1/3	0	-4	5/3	0	0	

most negative

$\Delta_1 = 10/3 - 1 = 1/3$ $\qquad\qquad$ $\Delta_4 = 5/3$

$\Delta_2 = 5 - 5 = 0$ $\qquad\qquad$ $\Delta_5 = 0$

$\Delta_3 = -4$ $\qquad\qquad$ $\Delta_6 = 0$

$R_2 \to \frac{R_2}{5}$, $R_3 \to R_3 - 4R_2$

	C_j		3	5	4	0	0	0	Ratio
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3	X_B / x_j
x_2	5	8/3	2/3	1	0	1/3	0	0	$\frac{8}{2} = 4$
x_3	4	14/15	-4/15	0	1	-2/15	1/5	0	$-\frac{14}{4}$ = -ve
S_3	0	89/15	$\boxed{41/15}$	0	0	-2/15	-4/5	1	$\frac{89}{41} = 2.$ → min ratio
Δ_j	Z = 256/15		-11/15	0	0	17/15	4/5	0	

$\Delta_1 = \frac{10}{3} - \frac{16}{15} - 3 = \frac{-11}{15}$ $\qquad\qquad$ $\Delta_4 = \frac{5}{3} - \frac{8}{15} = \frac{17}{15}$

$\Delta_2 = 5 - 5 = 0$ $\qquad\qquad$ $\Delta_5 = \frac{4}{5}$

$\Delta_3 = 4 - 4 = 0$ $\qquad\qquad$ $\Delta_6 = 0$

$R_3 \to R_3 \times \frac{15}{41}$, $R_2 \to R_2 + \frac{4}{15} R_3$, $R_1 \to R_1 - \frac{2}{3} R_3$

	c_j		3	5	4	0	0	0
B.V.	C_B	X_B	x_1	x_2	x_3	S_1	S_2	S_3
x_2	5	50/41	0	1	0	15/41	8/41	-10/41
x_3	4	62/41	0	0	1	-6/41	5/41	4/41
x_1	3	89/41	1	0	0	-2/41	-12/41	15/41
Δ_j	Z=765/41		0	0	0	45/41	24/41	11/41

$\Delta_1 = 3-3 = 0$

$\Delta_2 = 5-5 = 0$

$\Delta_3 = 4-4 = 0$

$\Delta_4 = \frac{75}{41} - \frac{24}{41} - \frac{6}{41} = \frac{45}{41}$

$\Delta_5 = \frac{40}{41} + \frac{20}{41} - \frac{36}{41} = \frac{24}{41}$

$\Delta_6 = -\frac{50}{41} + \frac{16}{41} + \frac{45}{41} = \frac{11}{41}$

$\Delta_j \geq 0$, the solution is optimum.

$$\boxed{\begin{array}{c} \text{Max } Z = \frac{765}{41} \\ x_1 = \frac{89}{41},\ x_2 = \frac{50}{41},\ x_3 = \frac{62}{41} \end{array}}$$

Ques 4:- Solve the following L.P.P. by two-phase method -

$$\text{Min } Z = x_1 + x_2$$

subject to: $2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

Sol: Converting the minimization into maximization

$$\text{Max } Z^* = (-Z) = -x_1 - x_2$$

subject to: $2x_1 + x_2 - S_1 + A_1 = 4$

$x_1 + 7x_2 - S_2 + A_2 = 7$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

where $S_1, S_2 \rightarrow$ Surplus variable

$A_1, A_2 \rightarrow$ Artificial variable

$x_1 = x_2 = S_1 = S_2$
$A_1 = 4$
$A_2 = 7$

Phase I – Assign a cost (-1) to Artificial variables and a cost (0) to all other variables.

$$Z' = 0x_1 + 0x_2 + 0S_1 + 0S_2 - A_1 - A_2$$

	C_j		0	0	0	0	-1	-1	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	A_1	A_2	C_B/x_j
A_1	-1	4	2	1	-1	0	1	0	4
A_2	-1	7	1	$\boxed{7}$	0	-1	0	1	1 → min ratio
Δ_j	$Z' = -11$		-3	-8	1	1	0	0	

$\Delta_1 = -2-1 = -3$ $\Delta_4 = 1$
$\Delta_2 = -1-7 = -8$ $\Delta_5 = 0$
$\Delta_3 = 1$ $\Delta_6 = 0$

$R_2 \rightarrow \frac{R_2}{7}$, $R_1 \rightarrow R_1 - R_2$

	C_j		0	0	0	0	-1	Ratio
B.V.	C_B	X_B	x_1	x_2	S_1	S_2	A_1	C_B/x_j
A_1	-1	3	$\boxed{13/7}$	0	-1	1/7	1	$\frac{21}{13} = 1.$ → min ratio
x_2	0	1	1/7	1	0	-1/7	0	7
Δ_j	$Z' = -3$		-13/7	0	1	-1/7	0	

$\Delta_1 = -13/7$ $\Delta_4 = -1/7$
$\Delta_2 = 0$ $\Delta_5 = 0$
$\Delta_3 = 1$

$R_1 \rightarrow R_1 \times \frac{7}{13}$, $R_2 \rightarrow R_2 - \frac{1}{7}R_1$

	c_j		0	0	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2
x_1	0	21/13	1	0	-7/13	1/13
x_2	0	10/13	0	1	1/13	-2/13
Δ_j	$Z'=0$		0	0	0	0

$\Delta_1 = 0$ $\Delta_3 = 0$

$\Delta_2 = 0$ $\Delta_4 = 0$

$$\Delta_j \geq 0$$

Phase II — $Z^* = -x_1 - x_2$

	c_j		-1	-1	0	0
B.V.	C_B	X_B	x_1	x_2	S_1	S_2
x_1	-1	21/13	1	0	-7/13	1/13
x_2	-1	10/13	0	1	1/13	-2/13
Δ_j	$Z^* = -31/13$		0	0	6/13	1/13

$\Delta_1 = -1+1 = 0$

$\Delta_2 = 0$

$\Delta_3 = \frac{7}{13} - \frac{1}{13} = \frac{6}{13}$

$\Delta_4 = -\frac{1}{13} + \frac{2}{13} = \frac{1}{13}$

$\Delta_j \geq 0$, the solution is optimum

$$\boxed{Z^* = -\frac{31}{13}, \quad Z = \frac{31}{13} \qquad x_1 = \frac{21}{13}, \quad x_2 = \frac{10}{13}}$$

Ques 5: Write the dual of the following problem —

$$\text{Min } Z = 2x_2 + 5x_3$$

subject to:

$$x_1 + x_2 \geq 2$$
$$2x_1 + x_2 + 6x_3 \leq 6$$
$$x_1 - x_2 + 3x_3 = 4$$
$$x_1, x_2, x_3 \geq 0$$

Sol: Changing into Standard Primal form -

$$x_1 + x_2 \geq 2$$
$$-2x_1 - x_2 - 6x_3 \geq -6$$
$$x_1 - x_2 + 3x_3 \geq 4$$
$$-x_1 + x_2 - 3x_3 \geq -4$$
$$x_1, x_2, x_3 \geq 0$$

Matrix form -

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & -6 \\ 1 & -1 & 3 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ -6 \\ 4 \\ -4 \end{bmatrix}$$

$$A x \geq b$$

$$\text{Mini } Z = (0 \ 2 \ 5)(x_1 \ x_2 \ x_3) = Cx$$

Duality -

$$\text{Max } Z_D = b'y$$
$$\text{Max } Z_D = [2 \ -6 \ 4 \ -4][y_1, y_2, y_3', y_3'']$$
$$= 2y_1 - 6y_2 + 4(y_3' - y_3'')$$
$$y_1, y_2, y_3', y_3'' \geq 0$$

$$A'y \leq C'$$

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 0 & -6 & 3 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3' \\ y_3'' \end{bmatrix} \leq \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

$$y_1 - 2y_2 + (y_3' - y_3'') \leq 0$$
$$y_1 - y_2 - (y_3' - y_3'') \leq 2$$
$$-6y_2 + 3(y_3' - y_3'') \leq 5$$

Let $(y_3' - y_3'') = y_3$

$$y_1 - 2y_2 + y_3 \leq 0$$
$$y_1 - y_2 - y_3 \leq 2$$
$$-6y_2 + 3y_3 \leq 5$$

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$$\text{Max } Z = 2y_1 - 6y_2 + 4y_3$$

Ans.

* Transportation Problem :-

Transportation problem is a special kind of linear programming problem in which goods are transported from a set of sources to a set of destination subject to supply and demand of the source and destination respectively such that the total cost of transportation problem is minimize.

Let n = The number of sources

m = The number of destination

a_i = The supply at the source i.

d_j = The demand at the destination j.

C_{ij} = The cost of transportation per unit from i^{th} source to j^{th} destination.

x_{ij} = The number of units to be transported from the source i^{th} to the j^{th} destination.

* Mathematical formulation of the Transportation Problem-

Mathematically, the problem may be stated as follows-

$$\text{Min}(Z) = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij}\, x_{ij}$$

Subject to : $\sum_{i=1}^{m} x_{ij} = a_i$ for $i = 1,2,3 \ldots m$

$$\sum_{j=1}^{n} x_{ij} = b_j \text{ for } j = 1,2,3 \ldots n$$

For a feasable solution to exist, it is necessary that total supply equals total requirement.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Destination

$$
\begin{array}{c|ccccccc|c}
 & W_1 & W_2 & \cdots & W_j & \cdots & W_n & & \text{Supply} \\
\hline
 & C_{11} & C_{12} & \cdots & C_{1j} & \cdots & C_{1n} & & a_1 \\
 & C_{21} & C_{22} & \cdots & C_{2j} & \cdots & C_{2n} & & a_2 \\
 & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
\text{Origin} & C_{i1} & C_{i2} & \cdots & C_{ij} & \cdots & C_{in} & & a_i \\
 & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\
 & C_{m1} & C_{m2} & \cdots & C_{mj} & \cdots & C_{mn} & & a_m \\
\hline
\text{Demand} & b_1 & b_2 & \cdots & b_j & \cdots & b_n & &
\end{array}
$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

* North-West corner cell method :-

Step 1- Find the minimum of the supply and demand value with respect to the current north-west corner cell of the cost matrix.

Step 2- Allocate the minimum value to the current north-west corner cell and subtract with minimum from the supply and demand value of the current north-west corner cell.

Step 3- Check whether exactly one of the row or column correspondence to north-west corner cell has 0 supply or demand respectively. If so, go to step 4 or step 5.

Step 4- Delete the row or column of the current north-west cell which has 0 supply or demand and go to step 6.

Step 5- Delete both the row and column with respect to the current north-west corner cell.

Step 6- Check whether exactly one row or column left out, if yes go to step 7 else go to step 1.

Step 7 - Match the supply or demand of the row or column with the remaining demands or supply of the undeleted columns or row.

Step 8 - Go to phase 2 for optimization of solution obtained above.

Ques -

Destination

		1	2	3	4	Supply
Source	1	3	1	7	4	300
	2	2	6	5	9	400
	3	8	3	3	2	500
	Demand	250	350	400	200	1200

Sol:

Destination

		1	2	3	4	Supply
Source	1	250 3	1	7	4	300/50
	2	2	6	5	9	400
	3	8	3	3	2	500
	Demand	250/0	350	400	200	1200

		2	3	4	Supply
Source	1	50 1	7	4	50/0
	2	6	5	9	400
	3	3	3	2	500
	Demand	350/300	400	200	950

Destination

		2	3	4	Supply
Source	2	300 6	5	9	400/100
	3	3	3	2	500
	Demand	300/0	400	200	900

Destination

Source		3	4	Supply
	2	$\boxed{100}$ 5	9	100 / 0
	3	3	2	500
Demand		400/300	200	600

Destination

Source		3	4	Supply
	3	$\boxed{300}$ 3	$\boxed{200}$ 2	500/200/0
Demand		300/0	200/0	500

Minimum cost $= 250 \times 3 + 1 \times 50 + 300 \times 6 + 100 \times 5 + 300 \times 3 + 200 \times 2$

$= 750 + 50 + 1800 + 500 + 900 + 400$

$= 4400$

Feasable solution —

$x_{11} = 250$ $\qquad$ $x_{23} = 100$

$x_{12} = 50$ $\qquad$ $x_{33} = 300$

$x_{22} = 300$ $\qquad$ $x_{34} = 200$

Ques —

Wardhouse

Factory	E	F	G	Available
A	10	8	9	15
B	5	2	3	20
C	6	7	4	30
D	7	6	8	35
Demand	25	26	49	100

Sol:

Wardhouse

	E	F	G	
A	$\boxed{15}$ 10	8	9	15 / 0
B	$\boxed{10}$ 5	$\boxed{10}$ 2	3	20/10/0
C	6	$\boxed{16}$ 7	$\boxed{14}$ 4	30 / 14
D	7	6	$\boxed{35}$ 8	35/0
	25/10/0	26/16/0	49/35/0	100

min cost $= 15 \times 10 + 10 \times 5 + 10 \times 2 + 16 \times 7 + 14 \times 4 + 35 \times 8$

$= 150 + 50 + 20 + 112 + 56 + 280$

$= 668$

Feasable solution-

$x_{AE} = 15$ $x_{CF} = 16$

$x_{BE} = 10$ $x_{CG} = 14$

$x_{BF} = 10$ $x_{DG} = 35$

Ques-

Destination

		1	2	3	4	Supply
Source	1	5	10	4	5	10
	2	6	8	7	5	25
	3	4	2	5	7	20
	Demand	25	10	15	5	55

Sol:

Destination

		1	2	3	4	Supply
Source	1	[10] 5	10	4	5	10/0
	2	[15] 6	[10] 8	7	5	25/10/0
	3	4	2	[15] 5	[5] 7	20/5/0
	Demand	25/15/0	10/0	15/0	5/0	55

Min cost $= 10 \times 5 + 15 \times 6 + 10 \times 8 + 15 \times 5 + 7 \times 5$

$= 50 + 90 + 80 + 75 + 35$

$= 330$

Feasable solution-

$x_{11} = 10$ $x_{33} = 15$

$x_{21} = 15$ $x_{34} = 5$

* Least Cost Cell Method :-

Ques - Solve by least cost cell method the following transportation problem -

	1	2	3	4	Supply
1	3	1	7	4	300
2	2	6	5	9	400
3	8	3	3	2	500
Demand	250	350	400	200	1200

Sol:

	1	2	3	4	
1	3	[300] 1	7	4	300/0
2	[250] 2	[50] 6	[100] 5	9	400/150/0
3	8	3	[300] 3	[200] 2	500/300/0
	250/0	350/50	400/100/0	200/0	1200

Minimum cost $= 300\times1 + 250\times2 + 50\times6 + 100\times5 + 300\times3 + 200\times2$

$= 300 + 500 + 300 + 500 + 900 + 400$

$= 2900$ Rs.

Initial basic feasible solution-

$x_{12} = 300$ $\qquad$ $x_{23} = 100$

$x_{21} = 250$ $\qquad$ $x_{33} = 300$

$x_{22} = 50$ $\qquad$ $x_{34} = 200$

Ques-

	D_1	D_2	D_3	D_4	
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
	5	8	7	14	34

Sol:

	D_1	D_2	D_3	D_4	Supply
Q_1	19	30	50	[7] 10	7/0
Q_2	[2] 70	30	[7] 40	60	9/2/0
Q_3	[3] 40	[8] 8	70	[7] 20	18/10/3/0
Demand	5/2/0	8/0	7/0	14/7/0	

Min cost $= 7\times10 + 2\times70 + 7\times40 + 3\times40 + 8\times8 + 7\times20$

$= 70 + 140 + 280 + 120 + 64 + 140$

$= 814$ Rs. Ans.

★ **Vogal's Approximation method (VAM method) or Penatly method –**

Ques–

	D_1	D_2	D_3	D_4	Supply
Q_1	19	30	50	10	7
Q_2	70	30	40	60	9
Q_3	40	8	70	20	18
Demand	5	8	7	14	

Sol:

	D_1	D_2	D_3	D_4	Supply	Penalty
Q_1	19	30	50	10	7	9
Q_2	70	30	40	60	9	10
Q_3	40	[8] 8	70	20	18	12
Demand	5	8/0	7	14		
Penalty	21	22 (max)	10	10		

	D_1	D_3	D_4	Supply	Penalty
Q_1	[5] 19	50	10	7/2	9
Q_2	70	40	60	9	20
Q_3	40	70	20	10	20
Demand	5/0	7	14		
Penalty	21 (max)	7	10		

	D_3	D_4	Supply	Penalty
O_1	50	10	2	40
O_2	40	60	9	20
O_3	70	$\boxed{10}$ 20	10/0	50 ← (max)
Demand	7	14/4		
Penalty	10	10		

	D_3	D_4	supply	Penalty
O_1	50	$\boxed{2}$ 10	2/0	40
O_2	40	60	9	20
Demand	7	4/2		
Penalty	10	50 (max)		

	D_3	D_4	Supply
O_2	$\boxed{7}$ 40	$\boxed{2}$ 60	9/2/0
Demand	7/0	2/0	

$$\text{Minimum cost} = 8 \times 8 + 5 \times 19 + 10 \times 20 + 2 \times 10 + 7 \times 40 + 2 \times 60$$
$$= 64 + 95 + 200 + 20 + 280 + 120$$
$$= 779 \text{ Rs.}$$

Ques – Solve by VAM method –

	D_1	D_2	D_3	Supply
I	10	8	9	15
II	5	2	3	20
III	6	7	4	30
IV	7	6	8	35
Demand	25	26	49	

	D_1	D_2	D_3	Supply	Penalty
I	10	8	9	15	1
II	5	$\boxed{20}$ 2	3	20/0	1
III	6	7	4	30	2
IV	7	6	8	35	1
Demand	25	26/6	49	100	
Penalty	1	4 (max)	1		

	D_1	D_2	D_3	Supply	Penalty
I	10	8	9	15	1
III	6	7	$\boxed{30}$ 4	30/0	2
IV	7	6	8	35	1
Demand	25	6	49/19	80	
Penalty	1	2	4 (max)		

	D_1	D_2	D_3	Supply	Penalty
I	10	8	9	15	1
IV	$\boxed{25}$ 7	6	8	35/10	1
Demand	25/0	6	19	50	
Penalty	3 (max)	2	1		

	D_2	D_3	Supply	Penalty
	8	$\boxed{15}$ 9	15	1
	$\boxed{6}$ 6	$\boxed{4}$ 8	10/4/0	2
Demand	6/0	19/15	25	
Penalty	2 (max)	1		

$$\text{Min cost} = 2\times20 + 30\times4 + 7\times25 + 6\times6 + 15\times9 + 4\times8$$
$$= 40 + 120 + 175 + 36 + 135 + 32$$
$$= 538 \text{ Rs.}$$

Feasable solution - $x_{22} = 20$ $\quad x_{13} = 15$

$x_{33} = 30$ $\quad x_{43} = 4$

$x_{41} = 25$ $\quad x_{42} = 6$

* MODI Method :-

Step 1– Find the initial basic feasable solution by using any of three methods.

Step 2– Check the number of occupied cells. If these are less than $m+n-1$, there exists degeneracy and we introduce a very small positive assignment of $\epsilon (\epsilon \to 0)$ in suitable independent positions, so that the number of occupied cells is exactly equal to $m+n-1$.

Step 3– For each occupied cell in the current solution, we solve the equation

$$c_{ij} = u_i + v_j$$

Starting initially with some $u_i = 0$ or $v_j = 0$.

Step 4– For all unoccupied cells

$$d_{ij} = c_{ij} - (u_i + v_j)$$

Step 5– If $d_{ij} \geq 0$ then the current basic feasable solution is an optimum one. If at least one $d_{ij} \leq 0$ select the unoccupied cell having the largest positive net evaluation to enter the basis.

Step 6– Let the unoccupied cell (r, s) enter the basis. Allocate an unknown quantity say θ to the cell (r, s). Identify a loop that starts and enters at the cell (r, s) and connects some of the basic cells. Add and subtract interchangeably θ to and from the transition cells of the loop.

Step 7 – Assign a maximum value θ in such a way that the value of one basic variable becomes zero and other basic variables remain non-negative. The basic cell whose allocation has been reduced to zero leaves the basis.

Step 8 – Return the step 3 and repeat the process until an optimum basic feasable solution has been obtained.

Ques –

19	30	50	10	7
70	30	40	60	9
70	8	70	20	10
5	8	7	14	34

$\boxed{5}$ 19	30	50	$\boxed{2}$ 10	7	u_1
70	30	$\boxed{7}$ 40	$\boxed{2}$ 60	9	u_2
70	$\boxed{8}$ 8	70	$\boxed{10}$ 20	10	u_3
5	8	7	14	34	
v_1	v_2	v_3	v_4		

$m+n-1 = 3+4-1 = 6 =$ Occupied cell

Calculate for occupied cells –

$C_{ij} = u_i + v_j$

$C_{11} = u_1 + v_1 \Rightarrow 19 = u_1 + v_1$

$C_{14} = u_1 + v_4 \Rightarrow 10 = u_1 + v_4$

$C_{23} = u_2 + v_3 \Rightarrow 40 = u_2 + v_3$

$C_{24} = u_2 + v_4 \Rightarrow 60 = u_2 + v_4$

$C_{32} = u_3 + v_2 \Rightarrow 8 = u_3 + v_2$

$C_{34} = u_3 + v_4 \Rightarrow 20 = u_3 + v_4$

Put $v_4 = 0$

$u_3 = 20$	$v_1 = 9$
$u_2 = 60$	$v_3 = -20$
$u_1 = 10$	$v_2 = -12$

Calculate for non-occupied cells -

$d_{ij} = C_{ij} - (u_i + v_j) \Rightarrow$

$d_{12} = 30 - (u_1 + v_2) \Rightarrow 30 - (10 - 12) = 32$

$d_{13} = 50 - (u_1 + v_3) \Rightarrow 50 - (10 - 20) = 60$

$d_{21} = 70 - (u_2 + v_1) \Rightarrow 70 - (60 + 9) = 1$

$d_{22} = 30 - (u_2 + v_2) \Rightarrow 30 - (60 - 12) = -18$

$d_{31} = 70 - (u_3 + v_1) \Rightarrow 70 - (20 + 9) = 41$

$d_{33} = 70 - (u_3 + v_3) \Rightarrow 70 - (20 - 20) = 70$

$d_{22} \leq 0$, This is not optimum

5			2
	2 +	7	− 2 10
	6 8 −		+ 10 12

Add 2 to (+ve) sign and subtract at (-ve) sign

5 19	30	50	2 10	7	u_1
70	2 30	7 40	60	9	u_2
70	6 8	70	12 20	18	u_3
5	8	7	14	34	
v_1	v_2	v_3	v_4		

Calculate by occupied cells -

$C_{ij} = u_i + v_j$

$C_{11} = u_1 + v_1 \Rightarrow 19 = u_1 + v_1$

$C_{14} = u_1 + v_4 \Rightarrow 10 = u_1 + v_4$

$C_{22} = u_2 + v_2 \Rightarrow 30 = u_2 + v_2$

$C_{23} = u_2 + v_3 \Rightarrow 40 = u_2 + v_3$

$C_{32} = u_3 + v_2 \Rightarrow 8 = u_3 + v_2$

$C_{34} = u_3 + v_4 \Rightarrow 20 = u_3 + v_4$

Put $u_1 = 0$

$v_1 = 19$	$u_2 = 32$
$v_4 = 10$	$v_3 = 8$
$v_2 = -2$	$u_3 = 10$

Calculate for non-occupied cells-

$d_{ij} = C_{ij} - (u_i + v_j)$

$d_{12} = 30 - (u_1 + v_2) \Rightarrow 30 - (0 - 2) = 32$

$d_{13} = 50 - (u_1 + v_3) \Rightarrow 50 - (0 + 8) = 42$

$d_{21} = 70 - (u_2 + v_1) \Rightarrow 70 - (32 + 9) = 19$

$d_{24} = 60 - (u_2 + v_4) \Rightarrow 60 - (32 + 10) = 18$

$d_{31} = 70 - (u_3 + v_1) \Rightarrow 70 - (10 + 19) = 41$

$d_{33} = 70 - (u_3 + v_3) \Rightarrow 70 - (10 + 8) = 52$

$d_{ij} \geq 0$ optimum solution

New initial basic feasable solution

$x_{11} = 5$ $x_{23} = 7$

$x_{14} = 2$ $x_{32} = 6$

$x_{22} = 2$ $x_{34} = 12$

Ques- Find the optimal solution of transportation problem-

	A	B	C	Available
I	50	30	220	1
II	90	45	170	3
III	250	200	50	4
Requirement	4	2	2	8

	A	B	C	Available	Penalty
I	50	30	220	1	20
II	90	45	170	3	45
III	250	200	[2] 50	4/2	150 (max)
Requirement	4	2	2/0	8	
Penalty	40	15	120		

	A	B	Available	Penalty
I	50	30	1	20
II	90	45	3	45
III	250	[2] 200	2/0	50(max)
Requirement	4	2/0	6	
Penalty	40	15		

	A	Available
I	[1] 50	1/0
II	[3] 90	3/0
	4/3/0	4

	A	B	C	
I	$\boxed{1}$ 50	30	220	1
II	$\boxed{3}$ 90	45	170	3
III	250	$\boxed{2}$ 200	$\boxed{2}$ 50	4
	4	2	2	8

$m + n - 1 = 3 + 3 - 1 = 5$

* Assignment problem :-

The assignment problem can be stated in the form of $(n \times n)$ square cost matrix of real number. Suppose there are n jobs and n persons are available for doing these jobs. Assume that each person can do each job at a time.

Let C_{ij} is the cost if the i^{th} person is assign the j^{th} job. The problem is to find an assignment that is which job should be assign to which person, so that the total cost for performing all jobs is minimum.

Jobs

$$\begin{array}{ll|cccccc|}
 & & 1 & 2 & 3 \cdots & j & \cdots n \\
\hline
 & 1 & C_{11} & C_{12} & C_{13} \cdots & C_{1j} & \cdots C_{1n} \\
 & 2 & C_{21} & C_{22} & C_{23} \cdots & C_{2j} & \cdots C_{2n} \\
\text{Persons} & 3 & C_{31} & C_{32} & C_{33} \cdots & C_{3j} & \cdots C_{3n} \\
 & i & C_{i1} & C_{i2} & C_{i3} \cdots & C_{ij} & \cdots C_{in} \\
 & n & C_{n1} & C_{n2} & C_{n3} \cdots & C_{nj} & \cdots C_{nn} \\
\hline
\end{array}$$

Mathematical formulation of Assignment Problem -

Mathematically, the assignment problem can be stated as-

The total minimum cost

$$\sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} x_{ij}$$

where $i = 1, 2, 3 \ldots n$

$j = 1, 2, 3 \ldots n$

Subject to restrictions

$$x_{ij} = \begin{cases} 1, & \text{if } i^{th} \text{ person assigned } j^{th} \text{ job.} \\ 0, & \text{if not.} \end{cases}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ (one job is done by the } i^{th} \text{ person, } i = 1,2,3 \ldots n)$$

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ (only one person should be assigned the } j^{th} \text{ job, } j = 1,2,3 \ldots n)$$

where x_{ij} denotes the j^{th} job is to be assigned to the i^{th} person.

* Difference between Assignment and Transportation Problem—

1- Assignment is the square matrix but transportation is not necessary square matrix if we put $m=n$ in transportation then it becomes the problem of assignment. Hence we say that assignment problem is the special case of transportation problem.

2- The numerical evaluations of such association are called 'effectiveness' instead of 'transportation costs'.

3- Mathematically, all a_i and b_j are unity and each x_{ij} is limited to one of the two values 0 and 1. In such circumstances, exactly n of the x_{ij} can be non-zero. One of each origin and one for each destination.

Ques—

Jobs

	1	2	3	4	5
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

Sol:

$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline A & 7 & 3 & 1 & 5 & 0 \\ B & 0 & 9 & 5 & 5 & 4 \\ C & 1 & 6 & 7 & 0 & 4 \\ D & 4 & 3 & 1 & 0 & 3 \\ E & 4 & 0 & 3 & 4 & 0 \\ \hline \end{array}$$

$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline A & 7 & 3 & \cancel{0} & 5 & \boxed{0} \\ B & \boxed{0} & 9 & 4 & 5 & 4 \\ C & 1 & 6 & 6 & \boxed{0} & 4 \\ D & 4 & 3 & \boxed{0} & \cancel{0} & 3 \\ E & 4 & \boxed{0} & 2 & 4 & \cancel{0} \\ \hline \end{array}$$

$A \to 5,\ B \to 1,\ C \to 4,\ D \to 3,\ E \to 2$

Minimum cost $= 1+0+2+1+5$

$= 9$ Ans.

Ques- Solve the assignment problem –

$$\begin{array}{c|cccc|} & I & II & III & IV \\ \hline A & 2 & 3 & 4 & 5 \\ B & 4 & 5 & 6 & 7 \\ C & 7 & 8 & 9 & 8 \\ D & 3 & 5 & 8 & 4 \\ \hline \end{array}$$

Sol:- Step 1 – Subtract the smallest element of each row from every element of the corresponding row in the given matrix.

$$\begin{array}{c|cccc|} & I & II & III & IV \\ \hline A & 0 & 1 & 2 & 3 \\ B & 0 & 1 & 2 & 3 \\ C & 0 & 1 & 2 & 1 \\ D & 0 & 2 & 5 & 1 \\ \hline \end{array}$$

Step 2- Subtract the smallest element of each column from the every element of the corresponding column in the given matrix.

	I	II	III	IV
A	0	0	0	2
B	0	0	0	2
C	0	0	0	0
D	0	1	3	0

Step 3- Make the assignment.

	I	II	III	IV
A	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	2
B	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
C	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$
D	$\cancel{0}$	1	3	$\boxed{0}$

$A \to I$, $B \to II$, $C \to III$, $D \to IV$

Total minimum cost $= 2+5+9+4$

$= 20$ Ans.

Ques- Solve the assignment problem-

	M_1	M_2	M_3
J_1	8	7	6
J_2	5	7	8
J_3	6	8	7

Sol:- Step1- Subtract the smallest element of each row from every element of the corresponding row in the given matrix.

	M_1	M_2	M_3
J_1	2	1	0
J_2	0	2	3
J_3	0	2	1

	I	II	III	IV	V	
A	30	$\boxed{0}$	35	30	15	✓
B	15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$	
C	30	$\cancel{0}$	35	30	20	✓
D	$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
E	20	$\cancel{0}$	25	15	15	✓
		✓				

	I	II	III	IV	V
A	15	$\cancel{0}$	20	15	$\boxed{0}$
B	15	15	$\boxed{0}$	10	$\cancel{0}$
C	15	$\boxed{0}$	20	15	5
D	$\boxed{0}$	15	20	$\cancel{0}$	5
E	5	$\cancel{0}$	10	$\boxed{0}$	$\cancel{0}$

~~Men~~ A → V, B → III, C → II, D → I, E → IV

Minimum cost = 200 + 130 + 110 + 50 + 80

= 570 Rs. Ans.

Ques Solve the assignment problem—

	I	II	III	IV
a	18	24	28	32
b	8	13	17	19
c	10	15	19	22

Sol: Given problem is an unbalanced assignment problem so we add a dummy row.

	I	II	III	IV
a	18	24	28	32
b	8	13	17	19
c	10	15	19	22
d	0	0	0	0

Step 2—

	I	II	III	IV
a	0	6	10	14
b	0	5	9	11
c	0	5	9	12
d	0	0	0	0

Step 3—

	I	II	III	IV	
a	$\boxed{0}$	6	10	14	✓
b	$\cancel{0}$	5	9	11	✓
c	$\cancel{0}$	5	9	12	✓
d	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	
	✓				

Step 4 –

	I	II	III	IV	
a	$\boxed{0}$	1	5	9	✓
b	$\cancel{0}$	$\boxed{0}$	4	6	✓
c	$\cancel{0}$	$\cancel{0}$	4	7	✓
d	5	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	
	✓	✓			

Step 5 –

	I	II	III	IV
a	$\boxed{0}$	1	1	5
b	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
c	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	3
d	9	4	$\cancel{0}$	$\boxed{0}$

$a \to I,\ b \to II,\ c \to III,\ d - IV$

Minimum cost = 18 + 13 + 19 + 0

= 50 Rs. Ans.

Ques – Solve the assignment problem –

5	3	1	8
7	9	2	6
6	4	5	7
5	7	7	6

Sol: Step 1 –

4	2	0	7
5	7	0	4
2	0	1	3
0	2	2	1

Step 2 –

4	2	0	6
5	7	0	3
2	0	1	2
0	2	2	0

Step 3 –

4	2	$\boxed{0}$	6	✓
5	7	$\cancel{0}$	3	✓
2	$\boxed{0}$	1	2	
$\boxed{0}$	2	2	$\cancel{0}$	
		✓		

Step 4 –

2	$\boxed{0}$	$\cancel{0}$	4	✓
3	5	$\boxed{0}$	1	
2	$\cancel{0}$	3	2	✓
$\boxed{0}$	2	4	$\cancel{0}$	
	✓			

$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	3
3	7	$\boxed{0}$	1
$\boxed{0}$	$\cancel{0}$	1	$\cancel{0}$
$\cancel{0}$	4	4	$\boxed{0}$

Min cost = 3 + 2 + 6 + 6

= 17 Ans.